MATHEMATICS-I

B.E. I-SEMESTER

(Common to All Branches)

Contents

Preface

ABOUT THE SUBJECT

Mathematics is a language in which every symbol and every combination has specific meaning which can be known by applying logical rules. This is the most challenging and important subject in all the levels of student education. The mathematic skills are not only required in various work environments ranging from services to manufacturing, but they are also must if you want to continue your education

At the basic level, mathematics education begins with learning skills such as identifying different shapes, measuring space, counting things using operations such as addition, subtraction, multiplication and division. It also introduces time, which measures duration of a complete day from sunrise to sunset and sunset to sunrise.

Based on this ground knowledge of mathematics, more advanced concepts are developed. At the higher secondary level**,** courses introduced will briefly revise the mathematic fundamentals, which are studied at schooling level.

Mathematics-I course introduced in the 1^{st} year 1-Sem (R18) B.E will briefly revise the mathematical concepts, which are studied at the intermediate level. In this course, you will learn the new concepts such as ***Sequences and Series, Calculus of one Variable, Multivariable Calculus (Differentiation), Multivariable Calculus (Integration), Vector Calculus .***

ABOUT THE BOOK

An attempt has been made through this book to present both theoretical and problematic knowledge of **"Mathematics-I"**. This book covers the complete syllabus of the subject as prescribed by **Osmania University**. The content or matter is prepared as per examination point of view (i.e., for both internal and end examinations). The concepts or topics are explained in simple and easy language, so that even an average student can easily grasp the subject knowledge.

It is sincerely hoped that this book will satisfy the expectations of students and at the same time helps them to score maximum marks in exams.

Suggestions for improvement of the book from our esteemed readers will be highly appreciated and incorporated in our forthcoming editions.

For Further Queries or Complaints Mail us at, **support@siaedugroup.com**

Syllabus

UNIT-1

Sequences and Series

Sequences, Series, General properties of series, Series of positive terms, Comparison tests, Tests of convergence D'Alembert's ratio test, Cauchy's n^{th} root test, Raabe's test, Logarithmic test, Alternating series, Series of positive and negative terms, Absolute convergence and conditional convergence.

UNIT-2

Calculus of One Variable

Rolle's theorem, Lagrange's, Cauchy's mean value theorems, Taylor's series, Curvature, Radius of curvature, Circle of curvature, Envelope of a family of curves, Evolutes and involutes.

UNIT-3

Multivariable Calculus (Differentiation)

Functions of two variables, Limits and continuity, Partial derivatives, Total differential and differentiability, Derivatives of composite and implicit functions (Chain rule), Change of variables, Jacobian, Higher order partial derivaties, Taylor's series of functions of two variables, Maximum and minimum values of functions of two variables, Lagrange's method of undetermined multipliers.

UNIT-4

Multivaraible Calculus (Integration)

Double integrals, Change of order of integration, Change of variables form Cartesian to plane polar coordinates, Triple integrals.

UNIT-5

Vector Calculus

Scalar and vector fields, Gradient of a scalar field, Directional derivative, Divergence and Curl of a vector field, Line, Surface and volume integrals, Green's theorem in a plane, Gauss's divergence theorem, Stoke's theorem (without proofs) and their verification.

FAQs & IQs FOR INTERNAL AND FINAL EXAMINATIONS

UNIT-1

SHORT QUESTIONS

Q1. Define limit.

Answer :

For answer refer Unit-1, Q.No. 2.

Q2. Define convergent sequence and give an example.

Answer :

For answer refer Unit-1, Q.No. 3.

Q3. Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$.

Answer : Dec.-16, Q3

For answer refer Unit-1, Q.No. 7.

Q4. State Leibnitz's test.

Answer : June/July-17, Q4

For answer refer Unit-1, Q.No. 14.

Q5. Define the terms (a) absolute convergence and (b) conditional convergence of a series with arbitrary terms.

Dec.-16, Q4

OR

Define the terms (a) absolute convergent series and (b) conditionally convergent series.

Answer : Dec.-17, Q4

For answer refer Unit-1, Q.No. 16.

ESSAY QUESTIONS

Q6. State and prove comparison test.

Answer :

For answer refer Unit-1, Q.No. 25.

Q7. Discuss the convergence of the series $\sum\left[\frac{\sqrt{n+1}-\sqrt{n}}{n^2}\right]$.

Answer : Dec.-16, Q17(a)

For answer refer Unit-1, Q.No. 30.

Q8. Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n,\ x>0$.

Answer : June/July-17, Q17(a)

For answer refer Unit-1, Q.No. 37.

Q9. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{1.4.7.....(3n-2)}{2.5.8.......(3n-1)}$.

Answer : Dec.-16, Q12(a)

For answer refer Unit-1, Q.No. 50.

Q10. Test the following ∞ series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ for conditional convergence.

Answer : Dec.-17, Q12(b)

For answer refer Unit-1, Q.No. 59.

Q11. Prove that the series $\sum(-1)^{n-1}\frac{\sin nx}{n^2}$ converges absolutely.

Answer : June/July-17, Q12(b)

For answer refer Unit-1, Q.No. 60.

Q12. Examine whether the series $-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+....$ is absolutely convergent or conditionally convergent.

Answer : Dec.-16, Q12(b)

For answer refer Unit-1, Q.No. 61.

UNIT-2

SHORT QUESTIONS

Q1. State Rolle's theorem.

Answer :

For answer refer Unit-2, Q.No. 1.

Q2. Find the c value of Rolle's mean value theorem for the function $f(x)=\log\left(\frac{x^2+ab}{x(a+b)}\right)$ on [a, b].

Answer :

For answer refer Unit-2, Q.No. 3.

Q3. Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x = 1$.

Answer :

For answer refer Unit-2, Q.No. 6.

Q4. Find the Taylor's series expansion of $f(x) = 2^x$ about x = 0.

Answer : Dec.-13, Q1

For answer refer Unit-2, Q.No. 11.

Q5. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point P(1, 1).

Answer : May/June-18, Q6

For answer refer Unit-2, Q.No. 12.

Q6. Find the radius of curvature at the origin of the curve $x^4 - 4x^3 - 18x^2 - y = 0$.

Answer : June/July-17, Q5

For answer refer Unit-2, Q.No. 13.

Q7. Obtain the equation of envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation ab = 4.

Answer : June/July-17, Q6

For answer refer Unit-2, Q.No. 23.

ESSAY QUESTIONS

Q8. Verify Lagrange's Mean Value Theorem for $f(x) = x(x-1)(x-2);\ x \in \left[0, \frac{1}{2}\right]$.

Answer : Dec.-17, Q13(a)

For answer refer Unit-2, Q.No. 35.

Q9. State Cauchy's mean value theorem and verify if for the functions $f(x) = e^{-x}$ and $g(x) = e^x$ in [a, b].

Answer : Dec.-16, Q13(a)

For answer refer Unit-2, Q.No. 36.

Q10. Find the circle of curvature of the curve $ay^2 = x^3$ at p(a, a).

Answer :

For answer refer Unit-2, Q.No. 48.

Q11. Determine envelope of one parameter family of curves of $f(x, y, \alpha) = 0$ where α is the parameter. Also find the envelope of the straight lines $x \cos\alpha + y \sin\alpha = l \sin\alpha \cos\alpha$, α being parameter.

Answer :

For answer refer Unit-2, Q.No. 54.

Q12. Show that the whole length of the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $4(a^2/b - b^2/a)$.

Answer :

For answer refer Unit-2, Q.No. 60.

UNIT-3

SHORT QUESTIONS

Q1. Evaluate $\lim\limits_{(x,y)\to(1,2)} \frac{x^2y}{x+y^2}$.

Answer : June/July-17, Q7

For answer refer Un it-3, Q.No. 1.

Q2. Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ doesn't exist.

Answer : Dec.-16, Q7

For answer refer Unit-3, Q.No. 2.

Q3. Determine $\underset{(x,y)\to(0,0)}{\text{Lt}} \frac{xy^2}{x^2+y^4}$ if it exists.

Answer : Dec.-15, Q5

For answer refer Unit-3, Q.No. 3.

Q4. Determine $\underset{(x,y)\to(1,-1)}{\text{Lim}} x^2 - y^2$.

Answer : April-16, Q5

For answer refer Unit-3, Q.No. 4.

Q5. Show that $f(x, y) = \begin{cases} \frac{x-y}{x+y}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$ is discontinuous at the point (0, 0).

Answer : Dec.-12, Q7

For answer refer Unit-3, Q.No. 5.

Q6. Find du/dt when $u = x^2 y$, $x = t^2$ and $y = e^t$.

Answer :

For answer refer Unit-3, Q.No. 12.

Q7. If $u = f(y - z, z - x, x - y)$, find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

Answer :

For answer refer Unit-3, Q.No. 15.

ESSAY QUESTIONS

Q8. If $f(x, y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$, show that $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ at (0, 0).

Answer : June/July-17, Q14(a)

For answer refer Unit-3, Q.No. 27.

Q9. Show that the function

$$f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$ **is not differentiable at (0, 0).**

Answer : Dec.-16, Q14(a)

For answer refer Unit-3, Q.No. 28.

Q10. Show that the function $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x-y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ **is not continuous at (0,0).**

June-13, Q14(b)

OR

Show that $f(x, y) = \begin{cases} \dfrac{x^2+y^2}{x-y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ **is not continuous at (0, 0).**

Answer : June-11, Q12(b)

For answer refer Unit-3, Q.No. 29.

Q11. Prove that $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1}(x) + \sin^{-1}(y)$ **are functionally dependent and find the relation between them.**

Answer :

For answer refer Unit-3, Q.No. 48.

Q12. Obtain the Taylor series expansion of the function $f(x, y) = e^{2x+y}$ **about (0, 0) upto third degree terms.**

Answer : June/July-17, Q14(b)

For answer refer Unit-3, Q.No. 55.

Q13. Find the absolute maximum and minimum values for the function $f(x, y) = x^2 - y^2 - 2y$ **in the closed region** $R : x^2 + y^2 \leq 1$.

Answer : Dec.-16, Q14(b)

For answer refer Unit-3, Q.No. 66.

Q14. Find the shortest and longest distances from the point (1, 2, – 1) to the sphere $x^2 + y^2 + z^2 = 24$.

Answer :

For answer refer Unit-3, Q.No. 70.

UNIT-4

SHORT QUESTIONS

Q1. Evaluate $\int_0^1\int_0^x e^{x+y}\,dy\,dx$

Answer :

For answer refer Unit-4, Q.No. 2.

Q2. Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$.

Answer :

For answer refer Unit-4, Q.No. 6.

Q3. Find the area bounded by the lines x = 0, y = 1 and y = x, using double integration.

Answer :

For answer refer Unit-4, Q.No. 11.

Q4. Find the area bounded by the lines x = 0, y = 1, x = 1 and y = 0.

Answer :

For answer refer Unit-4, Q.No. 12.

Q5. Change the order of integration in $\int_0^1\int_0^y f(x, y)\,dx\,dy$.

Answer :

For answer refer Unit-4, Q.No. 14.

ESSAY QUESTIONS

Q6. Find the values $\int\int xy\,dx\,dy$ **taken over the positive quadrant of the ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Answer :

For answer refer Unit-4, Q.No. 29.

Q7. Find the area of $r^2 = a^2\cos^2\theta$ **by double integration.**

Answer :

For answer refer Unit-4, Q.No. 30.

Q8. Using double integral find the area bounded by the parabolas $y^2 = 4ax$ **and** $x^2 = 4ay$.

Answer :

For answer refer Unit-4, Q.No. 35.

Q9. Change the order of integration for the given integral $\int_0^a\int_0^{2\sqrt{ax}} (x^2)\,dy\,dx$ **and evaluate it.**

Answer :

For answer refer Unit-4, Q.No. 39.

Q10. Evaluate by changing to polar coordinates

$$\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$$

Answer :

For answer refer Unit-4, Q.No. 49.

Q11. Transform the integral into polar coordinates and hence evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$.

Answer :

For answer refer Unit-4, Q.No. 53.

Q12. **Evaluate $\iiint_V \frac{dzdydx}{(x+y+z+1)^3}$ where V is the region bounded by x = 0, y = 0, z = 0 and x + y + z = 1.**

Answer :

For answer refer Unit-4, Q.No. 62.

Q13. **Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane z = 4.**

Answer :

For answer refer Unit-4, Q.No. 65.

UNIT-5

SHORT QUESTIONS

Q1. **Compute the gradient of the scalar function $f(x, y, z) = e^{xy} (x + y + z)$ at (2, 1,1).**

Answer : Dec.-16, Q9

For answer refer Unit-5, Q.No. 3.

Q2. **Find a unit normal vector to the surface $x^2 + y^2 + 2z^2 = 26$ at the point (2, 2, 3).**

Answer :

For answer refer Unit-5, Q.No. 9.

Q3. **If $\bar{f} = xy^2\,\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ then find div $\bar{f}$ at (1, –1, 1).**

Answer :

For answer refer Unit-5, Q.No. 11.

Q4. **If f is a differentiable scalar field, then show that $\nabla \times \nabla f = \vec{0}$.**

Answer : Dec.-16, Q10

For answer refer Unit-5, Q.No. 16.

Q5. **Show that the vector $e^{x+y-2z}(\hat{i} + \hat{j} + \hat{k})$ is solenoidal.**

Answer : June/July-17, Q10

For answer refer Unit-5, Q.No. 17.

Q6. **If $\bar{F} = (5xy - 6x^2)\,\bar{i} + (2y - 4x)\,\bar{j}$ then evaluate $\int \bar{F}.\overline{dR}$ along the curve $y = x^3$ from the point (1, 1) to (2, 8).**

Answer :

For answer refer Unit-5, Q.No. 26.

ESSAY QUESTIONS

Q7. **Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at (1, 2, 3) in the direction of the vector $2\hat{i} + 3\hat{j} + 6\hat{k}$.**

Answer : Dec.-17, Q17(b)

For answer refer Unit-5, Q.No. 38.

Q8. **Find the normal vector and unit normal vector to the surface $z^2 = x^2 - y^2$ at $(2, 1, \sqrt{3})$.**

Answer : June/July-17, Q9

For answer refer Unit-5, Q.No. 40.

Q9. **Show that the vector function $\vec{V} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ is irrotational and find its scalar potential.**

Answer : Dec.-16, Q15(a)

For answer refer Unit-5, Q.No. 53.

Q10. **If $\bar{F} = (5xy - 6x^2)\,\bar{i} + (2y - 4z)\,\bar{j}$, evaluate $\int_c \bar{F}.dr$, along the curve c in the xy - plane given by $y = x^3$ from the point (1, 1) to (2, 8).**

Answer : Dec.-17, Q15(b)

For answer refer Unit-5, Q.No. 55.

Q11. **Evaluate $\iint_S \vec{F}.\hat{n}ds$ where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$ and S is the portion of the plane 2x + 3y + 6z = 12 in the first octant.**

Answer : June/July-17, Q15(b)

For answer refer Unit-5, Q.No. 58.

Q12. **Use Green's theorem to evaluate the line integral $\oint_c (xy + x^2)\,dx + (x^2 + y^2)\,dy$, where C is the closed curve of the region bounded by y = x and $y = x^2$.**

Answer : Dec.-16, Q15(b)

For answer refer Unit-5, Q.No. 62.

LIST OF IMPORTANT FORMULAE

UNIT - 1

1. **Sequence**

A sequence is defined as an ordered set of real numbers such as, $u_1, u_2, u_3 \ldots u_n$.

2. **Limit**

A sequence tends to a limit 'l' if for every positive number ($\varepsilon > 0$), a value N of n can be obtained.

Such that,

$|u_n - l| < \varepsilon \; \forall \; n \geq N$

A limit can be represented as,

$$\underset{n \to \infty}{Lim}(u_n) = l$$

(or)

$$(u_n) \to l \text{ as } n \to \infty$$

3. **Convergent sequence**

A sequence is said to be convergent sequence if it has finite limit.

4. **Divergent Sequence**

A sequence which is not convergent is called as divergent sequence.

(or)

Let $\{s_n\}$ be a sequence. Then $\{s_n\}$ is said to be divergent sequence if $\underset{n \to \infty}{Lim}\, s_n = \pm\infty$ (or) $-\infty$.

5. **Comparison test**

Comparison Test-I

If Σu_n and Σv_n are the series of positive terms such that, $u_n \leq v_n \; \forall \; n$ and series Σv_n is convergent, then other series Σu_n is also convergent.

Comparison Test-II

If Σu_n and Σv_n are the series of positive terms such that, $u_n \geq v_n \; \forall \; n$ and series Σv_n is divergent, then Σu_n is is also divergent.

Comparison Test-III

If Σu_n and Σv_n are the series of positive terms such that $\lim_{n \to \infty} \frac{u_n}{v_n}$ = finite ($\neq 0$), then Σu_n and Σv_n both converge or diverge together.

6. **P-series test**

Let, $\Sigma \frac{1}{n^p}$ be a series, $P \in R$,

Then, $\Sigma \frac{1}{n^p}$

(i) Converges if $p > 1$

(ii) Diverges if $0 < p \leq 1$.

7. **D-Alembert's ratio test**

Let Σu_n be a series of positive terms such that $\underset{n\to\infty}{\text{Lim}} \frac{u_{n+1}}{u_n} = l$

Then Σu_n

(i) Converges if $l < 1$

(ii) Diverges if $l > 1$

(iii) Test fails if $l = 1$.

8. **State Raabe's test**

If Σu_n is a series of positive terms and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$.

Then,

(i) Series is convergent for $k > 1$

(ii) Series is divergent for $k < 1$

(iii) The test fails for $k = 1$.

9. **Cauchy's root test**

If Σu_n is a positive series such that $\lim_{n\to\infty} u_n^{1/n} = \lambda$ then,

(i) Σu_n is convergent if $\lambda < 1$

(ii) Σu_n is divergent if $\lambda > 1$

(iii) Test fails if $\lambda = 1$.

10. **Alternating Series**

A series which contains alternative positive and negative terms is known as alternating series. The expression for an alternating series is given as,

$u_1 - u_2 + u_3 - u_4 + \ldots\ldots. + (-1)^{n-1}\, u_n + \ldots\ldots$

(or)

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n$$

11. **Leibnitz's test for convergence of an alternating series**

If $\sum_{n=1}^{\infty} (-1)^{n-1}\, u_n$ is an alternating series then it is convergent if,

(a) $u_1 \geq u_2 > u_3 \geq \ldots\ldots \geq u_n \geq u_{n+1} \ldots$

(b) $\lim_{n\to\infty} u_n = 0$.

12. Absolute Convergence

Let Σu_n be a series of positive and negative terms. Then Σu_n is said to be absolutely convergent if $|\Sigma u_n|$ is convergent.

13. Conditionally Convergent

Let Σu_n be a series of positive and negative terms, then Σu_n is said to be conditionally convergent if,

(i) Σu_n is convergent

(ii) $|\Sigma u_n|$ is divergent.

UNIT - 2

1. Rolle's theorem.

If $\phi(x)$ is any function in the closed interval $[a, b]$ such that,

(i) $\phi(x)$ is continuous in $[a, b]$

(ii) $\phi(x)$ is differentiable in (a, b)

(iii) $\phi(a) = \phi(b)$

Then there exists at least one point $x = c$ in (a, b) such that $a < c < b$ and $\phi'(c) = 0$.

2. Lagrange's mean value theorem

If $f(x)$ is a function defined in $[a, b]$ such that,

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is derivable in (a, b)

Then, there exist atleast one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

3. Cauchy's mean value theorem

If $f : [a, b] \to R$, $g : [a, b] \to R$ are such that,

(i) f and g are continuous on $[a, b]$

(ii) f and g are differentiable on (a, b) and

(iii) $g'(x) \neq 0 \ \forall x \in (a, b)$

Then, there exists a point $c \in (a, b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4. Taylor's theorem

When $f(x), f'(x), f''(x)$ derivatives are continuous in $[a, a + h]$ and the n^{th} derivative of x exists in $(a, a + h)$, then a number θ lies between 0 and 1, such that the following series represents Taylor's theorem with Lagrange's form remainder.

$$f(a + h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots \frac{h^n}{n!} f^n(a + \theta h)$$

Where,

$$\frac{h^n}{n!} f^n(a + \theta h) = R_n \text{ (Remainder)}$$

5. Radius of curvature for curve in parametric form

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\left[\frac{d^2y}{dt^2}\right] - \frac{dy}{dt}\left[\frac{d^2x}{dt^2}\right]}$$

6. Radius of curvature

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

7. Envelope

The curve which touches all the members of family of curves is defined as the envelope to that family of curves.

8. Expression for circle of curvature

$$(x - \overline{x})^2 + (y - \overline{y})^2 = (\rho)^2$$

9. Centre of curvature

$$X = x - \left[\frac{y_1(1 + (y_1)^2)}{y_2}\right]$$

$$Y = y + \left[\frac{1 + (y_1)^2}{y_2}\right]$$

UNIT - 3

1. Limit

A function $f(x, y)$ is said to have limit L as 'x' tends to 'a' and 'y' tends to 'b' i.e,

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

2. Continuity

A function $f(x, y)$ at a point (a, b) is said to be continuous, if it is continuous at each point such that,

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

3. Derivatives of Composite Function

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \Psi(t)$, then 'u' is called a composite function of two variables and is expressed as,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

4. Derivatives of Implicit Function

A function of the form $f(x, y) = C$ is known as implicit function.

Where,

C - Constant

5. Taylor's series for the functions of two variables

$$f(x, y) = f(a, b) + \left[\frac{(x-a)}{1!} f_x(a,b) + \frac{(y-b)}{1!} f_y(a,b)\right] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + (y-b)^2 f_{yy}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)] + \ldots\ldots$$

UNIT - 4

1. Double Integrals

The generalized integral of a definite integral to two dimensions is known as double integral.

Properties

(i) $\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$

(ii) $\iint_R k\,f\,dx\,dy = k\iint_R f\,dx\,dy$, where k is a constant

(iii) $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$

Where,

R_1, R_2 - Two distinct regions of R and $R_1 \cup R_2 = R$.

(iv) Mean value theorem for double integral

The region 'R' consists of atleast one point (x_0, y_0) such that,

$$\iint_{R_2} f(x,y)\,dx\,dy = f(x_0, y_0)A \quad \text{(for continuous '}f\text{' in } R)$$

Where,

A – Area.

2. Jacobian of the coordinate transformation

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

3. Triple Integrals

The generalized integration of a definite integral to three dimensions is known as triple integral. In this case, the definite integral of a single variable function is extended to a function of three variables.

UNIT - 5

1. Gradient

The gradient of scalar point function f is defined as,

$$\text{grad } f = \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

2. Divergence

The divergence of a continuously differentiable vector point function '*F*' can be defined as,

$$\text{div } F = \nabla.F = i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$$

3. Curl

The curl of a continuously differentiable vector point function '*F*' can be expressed as,

$$\text{Curl } F = \nabla \times F = i\times\frac{\partial F}{\partial x} + j\times\frac{\partial F}{\partial y} + k\times\frac{\partial F}{\partial z}$$

4. The laplacian operator (∇^2) is defined as,

$$\nabla.\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left(i\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)$$

5. Solenoidal

A function $\hat{F}$ is said to be solenoidal if it satisfies the condition,

$$\nabla.f = 0$$

6. Irrotational

$\overline{F}$ is irrotational if $\nabla\times\overline{F} = 0$

7. Angle between the surfaces

Let θ be the angle between ϕ_1 and ϕ_2, then,

$$\cos\theta = \frac{\overline{n_1}\,.\,\overline{n_2}}{|n_1||n_2|}$$

8. Green's theorem

If '*S*' represents a closed region in *xy* plane bounded by a simple closed curve '*C*' and if *M*, *N* are continuous functions of *x* and *y*, then,

$$\int_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

9. Stoke's theorem

Let '*S*' be a surface bounded by a closed non-intersecting curve *C*. If $\vec{F}$ is any differentiable vector point function, then

$$\oint \vec{F}.d\vec{r} = \int \text{curl } \vec{F}.\vec{N}\, dS$$

Where '*C*' is traversed in positive direction and $\vec{N}$ is outward drawn unit normal vector.

10. Gauss's divergence theorem

Gauss's divergence theorem states that, for a continuously differentiable vector function located in the region *E* bounded by the closed surface *S*, then,

$$\int_S F.NdS = \int_E div\ F\, dV$$

UNIT 1

SEQUENCES AND SERIES

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Define sequence with an example.

Answer :

A sequence is defined as an ordered set of real numbers such as, $u_1, u_2, u_3 \ldots u_n$.

Basically, a sequence is represented as "u_n".

Example

1, 3, 5, 7 .... $(2n - 1)$.

Q2. Define limit.

Answer :

A sequence tends to a limit 'l' if for every positive number ($\varepsilon > 0$), a value N of n can be obtained.

Such that,

$|u_n - l| < \varepsilon \; \forall \; n \geq N$

A limit can be represented as,

$$\lim_{n \to \infty} (u_n) = l$$

(or)

$$(u_n) \to l \text{ as } n \to \infty$$

Q3. Define convergent sequence and give an example.

Answer :

A sequence is said to be convergent sequence if it has finite limit.

Example

$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots \frac{1}{n^2}$.

Q4. Define divergent sequence with an example.

Answer :

Divergent Sequence

A sequence which is not convergent is called as divergent sequence.

(or)

Let $\{s_n\}$ be a sequence. Then $\{s_n\}$ is said to be divergent sequence if $\lim_{n \to \infty} s_n = \pm\infty$ (or) $-\infty$.

Example

Let $\{s_n\} = \{n^2\}$

$\Rightarrow \quad s_n = n^2$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} n^2$$

$$= +\infty$$

$\therefore \quad \{s_n\}$ is divergent.

Q5. Define the convergence of an infinite series.

Answer :

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if

$$\sum_{n=1}^{\infty} u_n = \sum_{m=1}^{n} u_m = \underset{n\to\infty}{\text{Lt}}\, S_n = l.$$

Where l is a finite value and S_n is the n^{th} partial sum of the series.

Q6. State the necessary condition for a positive series Σa_n to be convergent.

Answer :

The necessary condition for a positive series Σa_n to be convergent is,

$$\lim_{n\to\infty} a_n = 0$$

i.e, as $n \to \infty$, $a_n \to 0$.

Q7. Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$.

Answer : Dec.-16, Q3

Given series is,

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$$

Let,

$$u_n = \left(1+\frac{1}{n^2}\right)^{n^2}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2} = e$$

$$\Rightarrow \lim_{n\to\infty} u_n = e \neq 0$$

$\therefore$ Σu_n is divergent

Hence, the given series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$ diverges.

Q8. Discuss the convergence of the series $\Sigma \frac{n^2+1}{n^2}$

Answer :

Given that,

$$\sum \frac{n^2+1}{n^2}$$

Let,

$$u_n = \frac{n^2+1}{n^2}$$

Apply limits on both sides,

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2+1}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} u_n = 1$$

$\therefore$ By theorem, if $\lim_{n\to\infty} u_n \neq 0$, then $\sum u_n$ is divergent

$\therefore$ $\sum \frac{n^2+1}{n^2}$ is divergent.

Q9. State comparison test.

Answer :

Comparison Test-I

If Σu_n and Σv_n are the series of positive terms such that, $u_n \leq v_n$ $\forall$ n and series Σv_n is convergent, then other series Σu_n is also convergent.

Comparison Test-II

If Σu_n and Σv_n are the series of positive terms such that, $u_n \geq v_n$ $\forall$ n and series Σv_n is divergent, then Σu_n is is also divergent.

Comparison Test-III

If Σu_n and Σv_n are the series of positive terms such that $\lim_{n\to\infty}\frac{u_n}{v_n}$ = finite ($\neq 0$), then Σu_n and Σv_n both converge or diverge together.

Q10. Test the convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

Answer :

Given series is,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

Let,

$$u_n = \frac{1}{n^2+1}$$

$$v_n = \frac{1}{n^2}$$

From comparison test,

$$\frac{u_n}{v_n} = \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}}$$

$$\Rightarrow \frac{u_n}{v_n} = \frac{n^2}{n^2+1}$$

$$\Rightarrow \quad \frac{u_n}{v_n} = \frac{n^2}{n^2(1+1/n^2)}$$

$$\Rightarrow \quad \frac{u_n}{v_n} = \frac{1}{1+1/n^2}$$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{1}{1+1/n^2}$$

$$= \frac{1}{1+0}$$

$$= 1 \neq 0$$

But, $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent.

Hence, Σv_n is also convergent by comparison test.

Q11. State P-series test.

Answer :

Let, $\Sigma \frac{1}{n^p}$ be a series, $P \in R$,

Then, $\Sigma \frac{1}{n^p}$

(i) Converges if $p > 1$

(ii) Diverges if $0 < p \leq 1$.

Q12. Find whether the series $\sum \frac{1}{n\sqrt{n^2-1}}$ is convergent or divergent.

Answer :

Given series is,

$$\sum \frac{1}{n\sqrt{n^2-1}} \quad ... (1)$$

Applying integral test,

$f(x) = \frac{1}{x\sqrt{x^2-1}}$ is a decreasing sequence in [2, ∞]

$$\therefore \quad f(x) = \int_2^t \frac{1}{x\sqrt{x^2-1}} dx$$

$$= \sec^{-1} x \Big|_2^t$$

$$= [\sec^{-1} t - \sec^{-1} 2]$$

$$= \underset{t\to\infty}{Lt} [\sec^{-1} t - \sec^{-1} 2]$$

$$= \sec^{-1} \infty - \sec^{-1} 2$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \to \text{finite}$$

The given series,

$$\therefore \quad \Sigma \frac{1}{n\sqrt{n^2-1}} \text{ is a convergent series.}$$

Q13. State Raabe's test.

Answer :

If Σu_n is a series of positive terms and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$. Then,

(i) Series is convergent for $k > 1$

(ii) Series is divergent for $k < 1$

(iii) The test fails for $k = 1$.

Q14. State Leibnitz's test.

Answer : June/July-17, Q4

If $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is an alternating series then it is convergent if,

(a) $u_1 \geq u_2 > u_3 \geq ..... \geq u_n \geq u_{n+1} ...$

(b) $\lim_{n\to\infty} u_n = 0$.

Q15. Define an alternating series.

Answer :

Alternating Series

A series which contains alternative positive and negative terms is known as alternating series. The expression for an alternating series is given as,

$$u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n-1} u_n + ...$$

(or)

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n$$

Q16. Define the terms (a) absolute convergence and (b) conditional convergence of a series with arbitrary terms.

Dec.-16, Q4

OR

Define the terms (a) absolute convergent series and (b) conditionally convergent series.

Answer : Dec.-17, Q4

Absolute Convergence

Let Σu_n be a series of positive and negative terms. Then Σu_n is said to be absolutely convergent if $|\Sigma u_n|$ is convergent.

Conditionally Convergent

Let Σu_n be a series of positive and negative terms, then Σu_n is said to be conditionally convergent if,

(i) Σu_n is convergent

(ii) $|\Sigma u_n|$ is divergent.

Q17. Show that the series $\sum \frac{\sin nx}{n^2}$ converges absolutely.

Answer :

Given series is,

$$\sum \frac{\sin nx}{n^2}$$

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\therefore$ $\Sigma \frac{\sin nx}{n^2}$ is convergent if $\Sigma \left|\frac{\sin nx}{n^2}\right|$ converges.

As $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$ $\quad [\because |\sin nx| \leq 1]$

And as $\Sigma \frac{1}{n^2}$ is a power series with power = 2 which is > 1

$\therefore$ $\Sigma \frac{1}{n^2}$ is convergent, and hence $\Sigma \left|\frac{\sin nx}{n^2}\right|$ converges

$\therefore$ The given series $\Sigma \frac{\sin nx}{n^2}$ is absolute convergent.

Q18. Show that the series $\sum \frac{\sin nx}{n^3}$ converges absolutely.

Answer :

Given series is,

$$\sum \frac{\sin nx}{n^3}$$

By definition, a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

$$\sum \frac{\sin nx}{n^3} = \frac{|\sin x|}{1^3} + \frac{|\sin 2x|}{2^3} + \frac{|\sin 3x|}{2^3} + ...$$

$$|u_n| = \frac{|\sin nx|}{n^3}$$

Let,

$$v_n = \frac{1}{n^3}$$

Then,

$$= \frac{|\sin nx|}{n^3} \leq \frac{1}{n^3} \quad (\because |\sin nx| \leq 1)$$

Also,

$\Sigma \frac{1}{n^3}$ is a power series with power = 3 > 1.

$\therefore$ $\Sigma \frac{1}{n^3}$ is convergent and hence $\Sigma \frac{|\sin nx|}{n^3}$ converges.

$\therefore$ The given series $\Sigma \frac{|\sin nx|}{n^3}$ is absolute convergent.

Q19. Show that series $\sum \frac{\cos nx}{n^2}$ is absolutely convergent.

Answer :

Given series is,

$$\sum \frac{\cos nx}{n^2}$$

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\therefore$ $\sum \frac{\cos nx}{n^2}$ is convergent if $\sum \left|\frac{\cos nx}{n^2}\right|$ converges.

As $\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2}$

$\Sigma \frac{1}{n^2}$ is a power series with power '2', which is > 1.

$\therefore$ $\Sigma \frac{1}{n^2}$ is convergent, and hence $\sum \left|\frac{\cos nx}{n^2}\right|$ converges.

$\therefore$ The given series $\sum \frac{\cos nx}{n^2}$ is absolutely convergent.

Q20. Discuss the convergence of the series $\sum \frac{1}{n^2}$.

Answer :

Given that,

$$\sum \frac{1}{n^2}$$

Here,

$$\sum \frac{1}{n^P} = \sum \frac{1}{n^2}$$

$P = 2 > 1$ [$\because$ $P > 1$ the series is convergent by P-test]

$\therefore$ $\sum \frac{1}{n^P}$ is convergent by P-test

Hence, $\sum \frac{1}{n^2}$ is convergent by P-test.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

1.1 SEQUENCES, SERIES, GENERAL PROPERTIES OF SERIES, SERIES OF POSITIVE TERMS

Q21. Define the following terms,

(a) Sequence

(b) Limit

(c) Convergent sequence

(d) Divergent sequence

(e) Bounded sequence

(f) Monotonic sequence.

Answer :

(a) Sequence

For answer refer Unit-3, Q1.

(b) Limit

For answer refer Unit-3, Q2.

(c) Convergent Sequence

For answer refer Unit-3, Q3.

(d) Divergent Sequence

For answer refer Unit-3, Q4.

(e) Bounded Sequence

A sequence of the form $u_n < k$ (where k is any number) is known as bounded sequence.

(f) Monotonic Sequence

A sequence which increases or decreases with respect to $u_{n+1} \geq u_n$ (or) $u_{n+1} \leq u_n$ is termed as monotonic sequence.

Examples

(i) 1, 4, 7, 10, ...

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$

A sequence which satisfies both monotonic and bounded sequence is known as convergent sequence.

Q22. Define and explain about oscillatory sequence.

Answer :

Oscillatory Sequence

Let a_n be a sequence. If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, then a_n is called as an "oscillatory sequence".

There are two types of oscillatory sequences. They are,

(i) Oscillate finitely and

(ii) Oscillate infinitely.

(i) Oscillate Finitely

If a bounded sequence does not converge, then it is said to oscillate finitely.

For example,

Consider the sequence $\{(-1)^n\}$.

Let,

$$a_n = (-1)^n$$

It is a bounded sequence,

$$\Rightarrow \quad \underset{n \to \infty}{Lt}\, a_{2n} = \underset{n \to \infty}{Lt}\, (-1)^{2n} = 1$$

and

$$\Rightarrow \quad \underset{n \to \infty}{Lt}\, a_{2n+1} = \underset{n \to \infty}{Lt}\, (-1)^{2n+1} = -1$$

Where,

$n = 1, 2, 3, ... \infty$.

$\therefore$ For the given sequence $\underset{n \to \infty}{Lt} \{a_n\}$ does not exist.

$\therefore$ The given sequence does not converge. Hence, the given sequence oscillates finitely.

(ii) Oscillate Infinitely

If an unbounded sequence does not diverge, then it is said to oscillate infinitely.

For example,

Consider, the sequence $\{(-1)^n.n\}$.

Let,

$$a_n = (-1)^n.n$$

It is an unbounded sequence,

$$\Rightarrow \quad \underset{n \to \infty}{Lt}\, a_{2n} = \underset{n \to \infty}{Lt}\, (-1)^{2n}.2n$$

$$= \underset{n \to \infty}{Lt}\, 2n = +\infty$$

$$\underset{n \to \infty}{Lt}\, a_{2n} = +\infty$$

and

$$\underset{n \to \infty}{Lt}\, a_{2n+1} = \underset{n \to \infty}{Lt}\, (-1)^{2n+1}.2n+1$$

$$= \underset{n \to \infty}{Lt}\, -(2n+1)$$

$$= -\infty$$

$$\therefore \underset{n \to \infty}{Lt}\, a_{2n+1} = -\infty$$

$\therefore$ The given sequence does not diverge.

Hence, the given sequence oscillates infinitely.

Q23. Write about the following,

(i) General properties of series

(ii) Series of positive terms.

Answer :

(i) General Properties of Series

Property 1

The addition or removal of finite number of terms from series, does not have any affect on the nature of the series (i.e., convergence or divergence of an infinite series).

Property 2

For a series containing positive and negative terms and all the positive terms are convergent, then the series remains convergent. Moreover, the presence of negative terms does not affect the nature of series.

Property 3

The multiplication of finite number to the terms of infinite series does not affect the nature of the series.

(ii) Series of Positive Terms

An infinite series which contains all the positive terms after few particular terms is called series of positive terms.

Example: – 1 – 2 – 3 + 4 + 5 + 6 + 7 + 8 + ...

Condition for Convergence

The necessary condition for convergence of a positive term series Σu_n is given as,

$$\Sigma \underset{n\to\infty}{Lim}\, u_n = 0$$

Q24. Define the convergence of an infinite series. Show that the n[th] term of a convergent series tends to zero. Is the converse true?

Answer :

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if $\sum_{n=1}^{\infty} u_n = \sum_{m=1}^{\infty} u_m = \underset{n\to\infty}{\text{Lt}}\, S_n = l$, Where l is a finite value and unique and S_n is the n^{th} partial sum of the series.

If $\underset{n\to\infty}{\text{Lt}}\, S_n$ does not exist, then the series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

If the series Σu_n is convergent, then $\underset{n\to\infty}{\text{Lt}}\, u_n = 0$. It is only a necessary condition and not sufficient so, the converse is not true.

If $\sum_{n=1}^{\infty} u_n$ is a sequence and if S_n and S_{n-1} are its nth and $(n-1)^{th}$ partial sums.

Then, $S_n - S_{n-1} = u_n$

$$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, (S_n - S_{n-1}) = 1 - 1 = 0$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\, u_n = 0$$

$\therefore$ Converse is not true.

1.2 COMPARISON TESTS, TESTS OF CONVERGENCE D'ALEMBERT'S RATIO TEST

Q25. State and prove comparison test.

Answer :

Comparison Test I

If Σu_n and Σv_n are the series of positive terms such that, $u_n \leq v_n$ $\forall$ n and series Σv_n is convergent, then other series Σu_n is also convergent.

Proof

Consider two series of positive terms,

$$u_n = u_1 + u_2 + u_3 + ... + u_n \text{ and}$$

$$v_n = v_1 + v_2 + v_3 + ... + v_n$$

For $u_n \leq v_n$

$$\lim_{n\to\infty} u_n < \lim_{n\to\infty} v_n$$

Since, Σv_n is convergent, $\lim\limits_{n\to\infty} v_n$ is finite

Therefore, $\lim\limits_{n\to\infty} u_n$ also has a finite value and it is also a convergent series.

Comparison Test II

If Σu_n and Σv_n are the series of positive terms such that, $u_n \geq v_n$ $\forall$ n and series Σv_n is divergent, then Σu_n also divergent.

Proof

Consider two series of positive terms,

$$u_n = u_1 + u_2 + u_3 + ... + u_n \text{ and}$$

$$v_n = v_1 + v_2 + v_3 + ... + v_n$$

For $u_n \geq v_n$

$$\lim_{n\to\infty} u_n \geq \lim_{n\to\infty} v_n$$

Since, Σv_n is divergent, $\lim\limits_{n\to\infty} v_n = \infty$

Therefore, $\lim\limits_{n\to\infty} u_n = \infty$ and Σu_n also divergent

Comparison Test III

If Σu_n and Σv_n are the series of positive terms such that $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$ = finite ($\neq 0$), then Σu_n and Σv_n both converge or diverge together.

Since,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l$$

By the definition of limit,

$$\left| \frac{u_n}{V_n} - l \right| < \varepsilon \text{ for } n \geq m$$

Where,

ε – Positive number

(or)

$$-\varepsilon < \frac{u_n}{v_n} - l < \varepsilon \quad ; \quad \text{for } n \geq m$$

(or)

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \, ; \quad \text{for } n \geq m$$

By eliminating the first 'm' terms of both the series,

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \text{ for all } n \qquad \text{... (1)}$$

Thus, there exist two cases.

Case(i): When Σv_n is Convergent

If Σv_n is convergent, then,

$$\lim_{n\to\infty}(v_1 + v_2 + v_3 + ... + v_n) = k \qquad \text{... (2)}$$

From equation (1),

$$\frac{u_n}{v_n} < l + \varepsilon$$

$\Rightarrow \quad u_n < (1 + \varepsilon)\, v_n \quad$ For all n

$\therefore \quad \lim_{n\to\infty}(u_1 + u_2 + ... + u_n) < (l + \varepsilon) \lim_{n\to\infty}(v_1 + v_2 + ... + v_n) = (l + \varepsilon)k \quad$ [$\because$ From equation (2)]

$\therefore \quad \Sigma u_n$ is also convergent

Case(ii): When Σv_n is Divergent

If Σv_n is divergent then,

$$\lim_{n\to\infty}(v_1 + v_2 + v_3 + ... + v_n) = \infty \qquad \text{... (3)}$$

From equation (1),

$$l - \varepsilon < \frac{u_n}{V_n}$$

$\Rightarrow \quad u_n > (1 - \varepsilon)v_n \quad$ For all n

$\therefore \quad \lim_{n\to\infty}(u_1 + u_2 + ... + u_n) > (l - \varepsilon) \lim_{n\to\infty}(v_1 + v_2 + ... + v_n) \to \infty \quad$ [$\because$ From equation (3)]

Therefore, Σu_n is also divergent.

Q26. Test the series $\sum\left(\sqrt{n^4 + 1} - \sqrt{n^4 - 1}\right)$ for convergence.

Answer :

Given series is,

$$u_n = \sqrt{n^4 + 1} - \sqrt{n^4 - 1}$$

$$u_n = \left[\sqrt{n^4 + 1} - \sqrt{n^4 - 1}\right] \times \frac{\left[\sqrt{n^4 + 1} + \sqrt{n^4 - 1}\right]}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}}$$

$$u_n = \frac{\left(\sqrt{n^4 + 1}\right)^2 - \left(\sqrt{n^4 - 1}\right)^2}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}}$$

$$u_n = \frac{n^4+1-n^4+1}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$v_n = \frac{\text{Highest power of 'n' in numerator}}{\text{Highest power of 'n' in denominator}}$$

$$\Rightarrow \quad v_n = \frac{n^0}{\sqrt{n^4}}$$

$$\Rightarrow \quad v_n = \frac{1}{n^2} \Rightarrow P = 2 > 1$$

$\therefore \quad \sum v_n$ is convergent by *P*-Test

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2\left[\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right]}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{1+1} = \frac{2}{2} = 1 \neq 0$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} \neq 0$$

$\Rightarrow \quad \sum u_n, \sum v_n$ are convergent by limit comparison test.

$\therefore \quad \sum u_n = \sum \left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]$ is convergent.

Q27. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$

Answer :

Given series is,

$$\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$$

Case (i) : When x > 1

Comparing given series Σu_n with $\Sigma v_n = \Sigma x^{-n}$,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{x^n + x^{-n}} . x^n$$

$$= \lim_{n\to\infty} \frac{1}{(x^n + x^{-n})\, x^{-n}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{x^n}{x^n} + x^{-2n}}$$

$$= \lim_{n\to\infty} \frac{1}{1 + x^{-2n}}$$

$$= \frac{1}{1+0} = 1 \qquad [\because x^{-2n} = 0 \text{ as } n \to \infty]$$

Σv_n is convergent.

$\therefore \quad \Sigma u_n$ is also convergent.

Case (ii): When x < 1

Comparing the given series Σu_n with $\Sigma v_n = \Sigma x^n$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{x^n + x^{-n}} \cdot \frac{1}{x^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{x^{2n} + \frac{x^n}{x^n}}\right) = \lim_{n\to\infty} \left(\frac{1}{x^{2n}+1}\right)$$

$$= \frac{1}{0+1} = 1 \qquad [\because x^{2n} = 0 \text{ as } n \to \infty]$$

Σv_n is convergent.

$\therefore \quad \Sigma u_n$ is also convergent.

Case (iii): When x = 1

$$\Sigma u_n = \frac{1}{2} + \frac{1}{2} + ... + \infty$$

Which is divergent.

$\therefore \quad \Sigma u_n$ is divergent when $x = 1$ and convergent for $x < 1$ and $x > 1$.

Q28. Test for convergence of the series

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + ...$$

Answer :

Given series is,

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + ...$$

Let, $\quad u_n = \frac{n+1}{n^p}$ and $v_n = \frac{1}{n^{p-1}}$

Then,

$$\frac{u_n}{v_n} = \frac{n+1}{n^p}.n^{p-1}$$

$$= \frac{n\left(1+\frac{1}{n}\right)}{n^p}.n^{p-1}$$

$$= \frac{1+\frac{1}{n}}{n^{p-1}}.n^{p-1}$$

$$\therefore \quad \frac{u_n}{v_n} = 1 + \frac{1}{n}$$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)$$

$$= \left(1+\frac{1}{\infty}\right) = (1 + 0) = 1 \neq 0$$

$\therefore$ By comparison test Σu_n and Σv_n are convergent.

Q29. Test for convergence of $\sum \frac{(n+1)(n+2)}{n^3\sqrt{n}}$.

Answer :

Given series is,

$$u_n = \frac{(n+1)(n+2)}{n^3\sqrt{n}}$$

$$\Rightarrow \quad u_n = \frac{n\left(1+\frac{1}{n}\right)n\left(1+\frac{2}{n}\right)}{n^3\sqrt{n}}$$

$$= \frac{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^3\sqrt{n}}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n\sqrt{n}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^{3/2}}$$

Let, $v_n = \frac{1}{n^{3/2}} = \frac{1}{n^P}$ where $P = \frac{3}{2} > 1$

$\therefore$ v_n is convergent by P-test.

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^{3/2}}}{\frac{1}{n^{3/2}}}$$

$$= \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \text{ as } n \to \infty, \frac{1}{n} \to 0$$

$$= (1 + 0)(1 + 0) = 1 \text{ (finite)}$$

$\therefore$ By comparison test, u_n and v_n convergent or divergent together.

$\therefore$ v_n is convergent, $\frac{u_n}{v_n}$ is also convergent.

$\therefore$ The given series $\sum \frac{(n+1)(n+2)}{n^3\sqrt{n}}$ is also convergent.

Q30. Discuss the convergence of the series $\sum\left[\frac{\sqrt{n+1}-\sqrt{n}}{n^2}\right]$.

Answer : Dec.-16, Q17(a)

Given series is,

$$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^2}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{n^2} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{(\sqrt{n+1})^2-(\sqrt{n})^2}{n^2(\sqrt{n+1}+\sqrt{n})}$$

$$= \frac{n+1-n}{n^2\left(\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)\right)}$$

$$= \frac{1}{n^2.n^{\frac{5}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{n^{\frac{5}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$v_n = \frac{\text{Highest Power of 'n' in numerator}}{\text{Highest Power of 'n' in denominator}}$$

Highest power of n in denominator

$$= \frac{n^0}{n^{\frac{5}{2}}} = \frac{1}{n^{\frac{5}{2}}}$$

$$\therefore \quad v_n = \frac{1}{n^{\frac{5}{2}}}$$

By p-series, $p = \frac{5}{2} > 1$; Σv_n is convergent.

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{lt} \frac{\frac{1}{n^{\frac{5}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)}}{\frac{1}{n}^{\frac{5}{2}}}$$

$$= \underset{n\to\infty}{lt} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{1}{\sqrt{1+0}+1}$$

$$= \frac{1}{1+1} = \frac{1}{2} \neq 0$$

$\therefore$ Σu_n; Σv_n are convergent.

$\therefore$ $\frac{\sqrt{n+1}-\sqrt{n}}{n^2}$ is convergent.

Q31. Test for convergence of the series,

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+.....\infty.$$

Answer :

Given series is,

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+...\infty$$

i.e., $\Sigma u_n = \sum \frac{(2n+1)}{n(n+1)(n+2)}$

$$\Rightarrow \quad u_n = \frac{2n+1}{n(n+1)(n+2)} \quad ...(1)$$

Numerator is a linear factor and denominator is a polynomial of degree 3.

$$\therefore \quad \Sigma v_n = \sum \frac{n}{n^3} = \sum \frac{1}{n^2}$$

$$\Rightarrow \quad v_n = \frac{1}{n^2} \quad ...(2)$$

Dividing equation (1) by equation (2),

$$\Rightarrow \quad \frac{u_n}{v_n} = \frac{(2n+1)}{n(n+1)(n+2)} \times \frac{n^2}{1}$$

$$= \frac{n^2.n\left(2+\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\frac{u_n}{v_n} = \frac{2+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{2+0}{(1+0)(1+0)}$$

$$= \frac{2}{1} = 2 \neq 0$$

Therefore, Σu_n and Σv_n converge or diverge together by comparison test.

$\Sigma v_n = \Sigma \frac{1}{n^2}$ is a P-series with $P = 2 > 1$ $\quad \left(\because \Sigma v_n = \Sigma \frac{1}{n^p} \text{ is a } P\text{-series}\right)$

A P-series with $P > 1$ converges

$\therefore \quad \Sigma v_n$ is convergent

Similarly, Σu_n is also convergent.

Q32. State and prove D-Alembert's ratio test.

Answer :

If Σu_n is a series of positive terms such that,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$$

(i) If $\lambda < 1$, Σu_n is convergent

(ii) If $\lambda > 1$, Σu_n is divergent

(iii) If $\lambda = 1$, test fails.

Case (i) : When $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda < 1$

From the definition of a limit, a positive r(< 1) can be determined such that,

$\frac{u_{n+1}}{u_n} < r$; for all $n > m$

By eliminating the first 'm' terms, the series is given as,

$u_1 + u_2 + u_3 + ...$

Where the common ratio is given as,

$$\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r \,.....$$

Then, $u_1 + u_2 + u_3 + u_4 + ... \infty$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\times\frac{u_2}{u_1}+\frac{u_4}{u_3}\times\frac{u_3}{u_2}\times\frac{u_2}{u_1}+.....\infty\right) < (1 + r + r^2 + r^3 + ... \infty)$$

$$= \frac{u_1}{1-r} \qquad \left(\because \text{Sum of Geometric series, } S_n = \frac{a(1-r)^n}{1-r}\right)$$

$\therefore$ Σu_n is convergent.

Case (ii): Where $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda > 1$

From the definition of limit, 'm' value can be determined, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $\geq m$

By eliminating the first 'm' terms the series is given as,

$u_1 + u_2 + u_3 + ...$

Where,

$$\frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1 \,.....$$

$$\therefore \quad u_1 + u_2 + u_3 + u_4 + ... u_n = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}.\frac{u_2}{u_1}+.....\right) \geq u_1(1+1+1+.....n)$$

$$= nu_1$$

$$\therefore \lim_{n\to\infty} (u_1 + u_2 + ... + u_n) \geq \lim_{n\to\infty} (nu_1)$$

As $\lim\limits_{n\to\infty} (nu_1) \to \infty$, Σu_n is said to be a divergent series

Case (iii) : When $\lambda = 1$

Consider a series $\Sigma u_n = \Sigma \dfrac{1}{n^p}$

$$\lambda = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{1}{(n+1)^p} n^p\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^p\left(1+\frac{1}{n}\right)^p} n^p\right] = \frac{1}{\left(1+\frac{1}{\infty}\right)^p}$$

$$= \frac{1}{(1+0)^p} = \frac{1}{1} = 1$$

Therefore, the series $\sum \dfrac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p < 1$.

For $p = 1$, it is not possible to find the nature of the series thus test fails at $\lambda = 1$.

Q33. Examine the convergence of the series $\dfrac{1}{1^p} + \dfrac{x}{3^p} + \dfrac{x^2}{5^p} + \ldots + \dfrac{x^{n-1}}{(2n-1)^p} + \ldots$

Answer :

Given series is,

$$\frac{1}{1^P} + \frac{x}{3^P} + \frac{x^2}{5^P} + \ldots + \frac{x^{n-1}}{(2n-1)^P}$$

Consider,

$$\Sigma u_n = \Sigma \frac{x^{n-1}}{(2n-1)^p}$$

$$\Rightarrow \quad u_n = \frac{x^{n-1}}{(2n-1)^p} = \frac{x^{n-1}}{n^p\left(2-\frac{1}{n}\right)^p}$$

Then, $u_{n+1} = \dfrac{x^{n+1-1}}{(2(n+1)-1)^p} = \dfrac{x^n}{(2n+1)^p} = \dfrac{x^n}{n^p\left(2+\frac{1}{n}\right)^p}$

By D-Alembertz ratio test,

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{\dfrac{x^n}{n^p(2+1/n)^p}}{\dfrac{x^{n-1}}{n^p(2-1/n)^p}} = \frac{x.\left(2-\frac{1}{n}\right)^p}{\left(2+\frac{1}{n}\right)^p}$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{x\left(2-\frac{1}{n}\right)^p}{\left(2+\frac{1}{n}\right)^p} = \frac{x(2-0)^p}{(2+0)^p} = \frac{x(2^p)}{2^p} = x$$

$\therefore$ Σu_n converges when $x < 1$, diverges when $x > 1$ and fails when $x = 1$.

$\therefore$ By limit comparison test for $n = 1$.

$$v_n = \frac{1}{n^p}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \frac{\frac{1}{n^p(2-1/n)^p}}{\frac{1}{n^p}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{(2-1/n)^p}$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{(2-1/n)^p} = \frac{1}{2} \text{ (finite)}$$

Σu_n converges when $x \le 1$ and diverges when $x > 1$.

Q34. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

Answer :

Given that,

$$\sum u_n = \sum_{n=1}^{\infty} \frac{n^2}{3^n}$$

$$\Rightarrow \quad u_n = \frac{n^2}{3^n}$$

$$\Rightarrow \quad u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$$

By applying D-Alembert's ratio test,

$$\frac{u_{n+1}}{u_n} = \frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}}$$

$$= \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2}$$

$$= \frac{n^2\left(1+\frac{1}{n}\right)^2}{3^n.3} \cdot \frac{3^n}{n^2} = \frac{n^2\left(1+\frac{1}{n}\right)^2}{3n^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{\left(1+\frac{1}{n}\right)^2}{3}$$

Applying limit on both sides,

$$\underset{n\to\infty}{\text{Lim}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lim}} \frac{\left(1+\frac{1}{n}\right)^2}{3}$$

$$= \frac{(1+0)^2}{3} = \frac{1^2}{3} \quad \left(\because n\to\infty, \frac{1}{n}\to 0\right)$$

$$= \frac{1}{3} < 1$$

$$\therefore \quad \underset{n\to\infty}{\text{Lim}} \frac{u_{n+1}}{u_n} < 1$$

$\therefore$ By D'Alembert's ratio test, the given series $\sum u_n$ is convergent.

Q35. Test for convergence of $\sum \frac{x^n}{n(n-1)(n-2)}$.

Answer :

Given that,

$$\sum \frac{x^n}{n(n-1)\,(n-2)}$$

Let,

$$u_n = \frac{x^n}{n(n-1)\,(n-2)}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)(n)(n-1)}$$

Applying D-Alembert's ratio-test

$$\frac{u_{n+1}}{u_n} = \frac{\frac{x^{n+1}}{n(n-1)(n+1)}}{\frac{x^n}{n(n-1)(n-2)}}$$

$$= \frac{\frac{x}{n+1}}{\frac{1}{n-2}} = \frac{x\,(n-2)}{(n+1)}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{xn\left(1-\frac{2}{n}\right)}{n\left(1+\frac{1}{n}\right)} = x$$

$\therefore$ By D'Alembert's ratio test,

Σu_n converges when $x < 1$, diverges when $x > 1$ and fails when $x = 1$

When $x = 1$, by comparison test,

i.e., $u_n = \dfrac{1}{n(n-1)\,(n-2)}$

$v_n = \dfrac{1}{n^3} = \dfrac{1}{n^P}$; $P = 3 > 1$

$\therefore$ v_n is convergent by P-test.

$$\frac{u_n}{v_n} = \frac{\frac{1}{n(n-1)(n-2)}}{\frac{1}{n^3}} = \frac{\frac{1}{n^3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\frac{1}{n^3}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \frac{(1)(1)}{1} = 1 \text{ (finite)}$$

$\therefore$ v_n converges.

$\therefore$ $\Sigma u_n = \sum \dfrac{x^n}{n(n-1)\,(n-2)}$ converges when $x \le 1$ and diverges when $x > 1$.

Q36. Discuss the convergence of the exponential series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Answer :

Given series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Let the given series be,

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$u_n = \frac{x^{n-1}}{(n-1)!}; \quad u_{n+1} = \frac{x^n}{n!}$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{x^{n-1}}{(n-1)!} \cdot \frac{n!}{x^n}$$

$$= \frac{n}{|x|} \text{ for } x \neq 0$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n}{|x|} = \infty > 1 \ \forall\ x \neq 0$$

$\therefore$ By D-Alembert's ratio test, the series $\sum_{n=1}^{\infty} |u_n|$ is convergent for $x \neq 0$.

When $x = 0$ the given series is,

$1 + 0 + 0 + \ldots$ which is convergent.

Hence, $\sum_{n=1}^{\infty} |u_n|$ is convergent $\forall\ x$.

$\therefore$ The given series is absolutely convergent for all x.

Q37. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n,\ x > 0$.

Answer : June/July-17, Q17(a)

Given series is,

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$$

Let, $u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n = \sqrt{\frac{n}{n^2+1}} . x^n$

$$\Rightarrow \quad u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} x^{n+1}$$

$$= \sqrt{\frac{n+1}{(n+1)^2+1}} x^{n+1}$$

By applying D-Alembert's ratio test

$$\frac{u_n}{u_{n+1}} = \frac{\sqrt{\frac{n}{n^2+1}} x^n}{\sqrt{\frac{n+1}{(n+1)^2+1}} x^{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1} \frac{n^2+2n+2}{n^2+1}} . \frac{1}{x}$$

$$\Rightarrow \underset{n\to\infty}{Lim} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lim} \sqrt{\frac{1}{1+\frac{1}{n}} \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} . \frac{1}{x} = \frac{1}{x}(1) = \frac{1}{x}$$

$\therefore$ By D-Alembert's ratio test,

Σu_n converges if $\frac{1}{x} > 1$ i.e., $x < 1$

Σu_n diverges if $\frac{1}{x} < 1$ i.e., $x > 1$ and the ratio test fails when $x = 1$

When $x = 1$,

$$u_n = \sqrt{\frac{n}{n^2+1}} (1)^n$$

$$= \sqrt{\frac{n}{n^2+1}}$$

$$= \sqrt{\frac{n}{\left(1+\frac{1}{n^2}\right)n^2}}$$

$$= \sqrt{\frac{1}{n}\left(\frac{1}{1+n^2}\right)}$$

$$= \frac{1}{\sqrt{n}} \sqrt{\frac{1}{1+n^2}}$$

Let, $v_n = \frac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{Lim} \frac{u_n}{v_n} = \underset{n\to\infty}{Lim} \frac{\frac{1}{\sqrt{n}} \sqrt{\frac{1}{1+\frac{1}{n^2}}}}{\frac{1}{\sqrt{n}}} = \underset{n\to\infty}{Lim} \sqrt{\frac{1}{1+\frac{1}{n^2}}} = 1$$

$\therefore \quad \underset{n\to\infty}{Lim} \frac{1}{1+\frac{1}{n^2}} \neq 0$ i.e., it is a finite value.

According to comparison test, Σu_n and Σv_n converge or diverge together.

As $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ which is of the form $\Sigma \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$

$\therefore \quad \Sigma v_n$ diverges

$\Rightarrow \quad \Sigma u_n$ diverges

Therefore, the series converges when $x < 1$ and diverges when $x \geq 1$.

Q38. Test for convergence of the series,

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+......$$

Answer :

Given series is,

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+......$$

i.e., $$u_n = \frac{x^{2n-2}}{\sqrt{n}(n+1)} \quad ... (1)$$

Then, $$u_{n+1} = \frac{x^{2(n+1)-2}}{((n+1)+1)\sqrt{n+1}}$$

$$u_{n+1} = \frac{x^{2n+2-2}}{(n+2)\sqrt{n+1}}$$

$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \quad ... (2)$$

Dividing equation (2) by equation (1),

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}\cdot\frac{\sqrt{n}(n+1)}{x^{2n-2}}$$

$$= \frac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}(x^{2n+2-2n})$$

$$= \left(\frac{n+1}{n+2}\left(\frac{n}{n+1}\right)^{1/2}\right)x^2$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}\cdot\frac{n^{1/2}}{n^{1/2}\left(1+\frac{1}{n^{1/2}}\right)}\right]x^2 = x^2$$

Hence, Σu_n converges when $x^2 < 1$ and

Σu_n diverges when $x^2 > 1$

When,

$x^2 = 1$

$$u_n = \frac{1}{(n+1)\sqrt{n}}$$

$$= \frac{1}{(n+1)n^{1/2}}$$

$$= \frac{1}{n^{3/2}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)}$$

and,

$$v_n = \frac{1}{n^{3/2}}$$

$$\therefore \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{n^{3/2}}\cdot\frac{n^{3/2}}{\left(1+\frac{1}{n}\right)} = 1$$

$\therefore$ Σv_n is convergent series, and

Σu_n is also convergent.

$\therefore$ The given series converges when and $x^2 > 1$ it diverges, when $x^2 > 1$.

1.3 CAUCHY'S n^{th} ROOT TEST, RAABE'S TEST, LOGARITHMIC TEST

Q39. Write about Cauchy's root test.

Answer :

If Σu_n is a positive series such that $\lim\limits_{n\to\infty} u_n^{1/n} = \lambda$ then

(i) Σu_n is convergent if $\lambda < 1$

(ii) Σu_n is divergent if $\lambda > 1$

(iii) Test fails if $\lambda = 1$

Case (i): $\lambda < 1$

If $\lim\limits_{n\to\infty} u_n^{1/n} = \lambda < 1$

From the definition of limit, the value of a positive number r ($\lambda < r < 1$) can be determined, such that,

$(u_n)^{1/n} < r$ for all $n > m$

$\Rightarrow$ $u_n < r^n$ for all $n > m$

$\because$ $r < 1$ then Σr^n is convergent.

Hence, by comparison test, Σu_n is also convergent.

Case (ii): $\lambda > 1$

If $\lim\limits_{n\to\infty} (u_n)^{1/n} = \lambda > 1$

From the definition of limit a positive number 'm' can be determined, such that,

$(u_n)^{1/n} > 1$ for all $n > m$

(or)

$u_n > 1$ for all $n > m$

By eliminating the first m terms, the series is given as,

$u_1 + u_2 + u_3 + ...$ where $u_1 > 1, u_2 > 1, u_3 > 1$

$\therefore$ $u_1 + u_2 + u_3 + ... + u_n > n$

And $\lim\limits_{n\to\infty} (u_1 + u_2 + ... + u_n) \to \infty$

Thus, Σu_n is divergent.

Case (iii): $\lambda = 1$

When $\lambda = 1$ Cauchy's root test fails and other tests are to be applied.

Q40. Discuss the convergence of the series,

$$\sum \frac{[(n+1)x]^n}{n^{n+1}}.$$

Answer :

Given series is,

$$\sum \frac{[(n+1)x]^n}{n^{n+1}}$$

i.e., $$u_n = \frac{((n+1)x)^n}{n^{n+1}}$$

$$\underset{n\to\infty}{Lim}(u_n)^{1/n} = \underset{n\to\infty}{Lim}\left[\frac{[(n+1)x]^n}{n^n.n^1}\right]^{1/n}$$

$$= \underset{n\to\infty}{Lim}\frac{(n+1)x}{n^{1+1/n}}$$

$$= \underset{n\to\infty}{Lim}\frac{n\left(1+\frac{1}{n}\right)x}{n^{1+1/n}}$$

$$= \underset{n\to\infty}{Lim}\frac{n\left(1+\frac{1}{n}\right)x}{n^{1/n}}$$

$$= \frac{(1+0)x}{n^0} = x$$

$$\underset{n\to\infty}{Lim}(u_n)^{1/n} = x$$

By Cauchy's root test, Σu_n

(i) Converges, if $x < 1$

(ii) Diverges, if $x > 1$

(iii) At $x = 1$, Root test fails then

By applying limit comparison test,

$$u_n = \left(\frac{((n+1)x)^n}{n^{n+1}}\right)$$

$$u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n.n^1}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n}{n} \quad \text{... (1)}$$

$$v_n = \frac{1}{n} \quad \text{... (2)}$$

Dividing equation (1) by equation (2) and applying limit.

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\frac{\frac{\left(1+\frac{1}{n}\right)^n}{n}}{(1/n)}$$

$$= \underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{1}{n}\right)^n}{n}.n = e \neq 0$$

If $\underset{n\to\infty}{\text{Lt}}\, u_n \neq 0$ then Σu_n is divergent.

$\therefore$ Σu_n is divergent for $x = 1$ ... (2)

Combining equations (1) and (2),

Hence, Σu_n

(i) Converges if $x < 1$

(ii) Diverges if x ≥ 1.

Q41. Test the series $\sum\left(1+\frac{1}{n}\right)^n$ for convergence.

Answer :

Let, $$u_n = \left(1+\frac{1}{n}\right)^n \quad \text{... (1)}$$

Taking n^{th} root on both sides of equation (1),

$$\sqrt[n]{u_n} = \sqrt[n]{\left(1+\frac{1}{n}\right)^n}$$

$$\Rightarrow (u_n)^{1/n} = \left[\left(1+\frac{1}{n}\right)^n\right]^{1/n}$$

$$\Rightarrow (u_n)^{1/n} = 1+\frac{1}{n} \quad \text{... (2)}$$

Applying limits on both sides of equation (2),

$$\underset{n\to\infty}{\text{Lt}}(u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)$$

$$= 1+\frac{1}{\infty}$$

$$= 1+0 \qquad \left[\because \frac{1}{\infty}=0\right]$$

$$\underset{n\to\infty}{\text{Lt}}(u_n)^{1/n} = 1$$

Therefore, according to Cauchy's root test,

$$\left.\begin{array}{l}\Sigma u_n \text{ is convergent if } x<1 \text{ and}\\ \Sigma u_n \text{ is divergent if } x>1\end{array}\right\} \quad \text{... (3)}$$

Here, $x = 1$, Cauchy's root test fails.

Now, applying some other test such that,

$$u_n = \left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \quad u_n = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{-n}}\right] \qquad \dots (4)$$

Applying n^{th} root test on equation (4),

$$\Rightarrow \quad (u_n)^{1/n} = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{-n}}\right]^{1/n}$$

$$\Rightarrow \quad (u_n)^{1/n} = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{-n\left(\frac{1}{n}\right)}}\right]$$

$$\Rightarrow \quad (u_n)^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^{-1}} \qquad \dots (5)$$

Taking limits on both sides of equation (5),

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{n}\right)^{-1}}$$

$$= \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)$$

$$= 1+\frac{1}{\infty}$$

$$= 1 + 0$$

$$= 1$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} \neq 0$$

If $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} \neq 0$ then Σu_n is divergent.

Therefore, the given series is divergent.

Since, $\Sigma u_n = 1$ … (6)

Thus, from equations (3) and (6),

Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

Q42. Test for convergence of the series $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

Answer :

$$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$$(u_n)^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{\frac{-n^{3/2}}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}}$$

$$(u_n)^{1/n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\underset{n\to\infty}{Lt} (u_n)^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{(1+0)^{\sqrt{n}}} = \frac{1}{e} < 1.$$

$\therefore$ By Cauchy's n^{th} root test, Σu_n converges.

Q43. Test the convergence of the series $\left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \dots$

Answer :

Given series is,

$$\left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \dots$$

$$= \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)x^n$$

$$\text{Let, } u_n = \left(\frac{n+1}{n+2}\right)x^n$$

$$(u_n)^{1/n} = \left(\frac{n+1}{n+2}\right)^{1/n} .x$$

$$\text{Lt}_{n\to\infty} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{n^{1/n}\left(1+\frac{1}{n}\right)^{1/n}}{n^{1/n}\left(1+\frac{2}{n}\right)^{1/n}} x$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^{1/n}}{\left(1+\frac{2}{n}\right)^{1/n}} x = x$$

∴ By Cauchy's n^{th} root test, the series converges if $x < 1$, diverges if $x > 1$ ... (1)

For $x = 1$,

$$u_n = \left(\frac{n+1}{n+2}\right)$$

$$\underset{n\to\infty}{\text{Lt}}(u_n) = \underset{n\to\infty}{\text{Lt}}\left[\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right] = 1 \neq 0 \quad \text{... (2)}$$

∴ From equations (1) and (2), the given series converges if $x < 1$ and diverge if $x \geq 1$.

Q44. Discuss the convergence of the series $\sum\left(1+\frac{1}{n}\right)^n x^n, x > 0.$

Answer :

Given series is,

$$\sum\left(1+\frac{1}{n}\right)^n x^n, \; x > 0$$

Let,

$$u_n = \left(1+\frac{1}{n}\right)^n .x^n$$

$$\Rightarrow \quad (u_n)^{1/n} = \left[\left(1+\frac{1}{n}\right)^n .x^n\right]^{1/n}$$

$$\Rightarrow \quad (u_n)^{1/n} = \left(1+\frac{1}{n}\right).x$$

$$\underset{n\to\infty}{Lt}\; u_n^{1/n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)x$$

$$= x$$

$$\therefore \quad \underset{n\to\infty}{Lt}\; u_n^{1/n} = x \quad \text{... (1)}$$

From Cauchy's n^{th} root test, if Σu_n is a series of positive terms such that $\underset{n\to\infty}{Lt}\; u_n^{1/n} = p$, then,

(a) Σu_n converges if $p < 1$

(b) Σu_n diverges if $p > 1$ and

(c) Test fails to decide nature if $p = 1$

∴ From equation (1),

Σu_n is convergent if $x < 1$, divergent if $x > 1$ and test fails if $x = 1$

When x = 1

$$u_n = \left(1+\frac{1}{n}\right).x$$

$$\underset{n\to\infty}{Lt}\; u_n^{1/n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)x$$

$$= e \neq 0$$

∴ Σu_n is divergent

Hence the series is convergent if $x < 1$ and divergent if $x \geq 1$.

Q45. Discuss the convergence of the series $\sum\left(\frac{n+2}{n+3}\right)^n x^n$.

Answer :

Given series is,

$$\sum\left(\frac{n+2}{n+3}\right)^n x^n$$

Let,

$$u_n = \sum\left(\frac{n+2}{n+3}\right)^n x^n$$

$$\therefore \quad u_n^{1/n} = \left(\frac{n+2}{n+3}\right)^{\frac{n}{n}} x^{\frac{n}{n}}$$

$$= \left(\frac{n+2}{n+3}\right)x$$

$$= \frac{n}{n}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$\therefore \quad \lim_{n\to\infty} = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$= \left(\frac{1+0}{1+0}\right)x$$

$$= x$$

By Cauchy's n^{th} root test, Σu_n converges if $x < 1$ and diverges if $x > 1$ and fails if $x = 1$

For $x = 1$,

$$u_n = \left(\frac{n+2}{n+3}\right)^n$$

$$= \frac{n^n\left(1+\frac{2}{n}\right)^n}{n^n\left(1+\frac{3}{n}\right)^n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$$

$$= \frac{e^2}{e^3} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2 \quad \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3\right]$$

$$= \frac{1}{e} \neq 0$$

$\therefore$ Σu_n is not convergent.

Thus, Σu_n is divergent.

Therefore, the series is convergent if $x < 1$ and divergent if $x \geq 1$.

Q46. State and prove Raabe's test.

Answer :

Raabe's Test

If Σu_n is a series of positive terms, and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$. Then,

(i) Series is convergent for $k > 1$

(ii) Series is divergents for $k < 1$

(iii) The test fails for $k = 1$.

Proof

Consider the two series Σu_n and Σv_n

Where, $\Sigma v_n = \Sigma \frac{1}{n^p}$ and it is convergent for $p > 1$.

Case (i) : For k > 1

Assume a number p such that,

$k \geq p < 1$

Comparing Σu_n with Σv_n, which is convergent as $p > 1$,

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$$

i.e., $$\frac{u_n}{u_{n+1}} \geq \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\geq \left(\frac{n+1}{n}\right)^p$$

$$= \left(1+\frac{1}{n}\right)^p$$

$$= 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

[$\because$ From binomial theorem]

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 \geq \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \geq p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) \geq \lim_{n\to\infty}\left[p\frac{+(p-1)}{2!}\frac{1}{n} + \ldots\right]$$

If $k > p$ then Σu_n is convergent.

Case (ii) : For k < 1

Select a number p such that,

$k \leq p > 1$

Comparing Σu_n with Σv_n, which is divergent as $p < 1$,

$$\frac{u_n}{u_{n+1}} \leq \frac{v_n}{v_{n+1}}$$

$$\frac{u_n}{u_{n+1}} \leq \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\leq \left(\frac{n+1}{n}\right)^p$$

$$= \left(1+\frac{1}{n}\right)^p$$

$$= 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

[$\because$ From binomial theorem]

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 \leq \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots..$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots..$$

$$\underset{n\to\infty}{Lim}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq \underset{n\to\infty}{Lim}\left[p + \frac{p(p-1)}{2!}.\frac{1}{n} + \ldots...\right]$$

Therefore, if $k < 1$, then Σu_n is divergent.

Case (iii): For k = 1

If $k = 1$, the test fails because it is not possible to determine whether series is convergent or divergent.

Q47. Discuss the convergence of the series $\sum \frac{4.7.10.\ldots\ldots(3n+1)}{1.2.3.\ldots\ldots n} x^n$.

Answer :

Given series is,

$$\Sigma u_n = \sum \frac{4.7.10.\ldots\ldots(3n+1)}{1.2.3.\ldots\ldots n} x^n$$

$$\Rightarrow \quad u_n = \frac{4.7.10.\ldots\ldots(3n+1)}{1.2.3.\ldots\ldots n} x^n \quad \ldots (1)$$

$$\Rightarrow \quad u_{n+1} = \frac{4.7.10...........(3n+1)(3n+4)}{1.2.3...........n(n+1)} x^{n+1} \quad ... (2)$$

Dividing equation (2) by equation (1),

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{4.7.10.......(3n+1)(3n+4)}{1.2.3.............n(n+1)} x^{n+1} \times \frac{1.2.3...........n}{4.7.10.........(3n+1)} . \frac{1}{x^n}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+1)(3n+4)}{(n+1)(3n+1)x^n} x^{n+1}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+4)}{(n+1)} .x$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{n\left(3+\frac{4}{n}\right)}{n\left(1+\frac{1}{n}\right)} x$$

Applying limit on both sides,

$$\Rightarrow \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{\left(3+\frac{4}{n}\right)}{\left(1+\frac{1}{n}\right)} x$$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = 3x$$

(or)

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \frac{1}{3x}$$

By ratio test,

(i) Converges, if $3x < 1$

i.e., $x < \frac{1}{3}$

(ii) Diverges, if $3x > 1$

i.e., $x > \frac{1}{3}$

or

$\frac{1}{3} < x$

(iii) At $x = \frac{1}{3}$, ratio test fails

By applying Raabe's test.

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+4)}{(n+1)} x$$

$$= \frac{(3n+4)}{(n+1)} . \frac{1}{3}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+4)}{(3n+3)}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(3n+3)}{(3n+4)}$$

Substituting '1' on both sides

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{3n+3}{3n+4} - 1$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{3n+3-3n-4}{3n+4}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{-1}{3n+4}$$

Multiplying 'n' on both sides,

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-n}{(3n+4)} = \frac{-n}{n\left(3+\frac{4}{n}\right)}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-1}{3+\frac{4}{n}}$$

$$\underset{n\to\infty}{Lt} \; n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-1}{3} < 1$$

By Raabe's test the given series is divergent, for $x = \frac{1}{3}$

$\therefore$ The given series is convergent if $x < \frac{1}{3}$ and divergent if $x \geq \frac{1}{3}$.

Q48. Test for convergence $\frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \frac{3.6.9.12}{7.10.13.16}x^4 + ...$

Answer :

Given series is,

$$\frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \frac{3.6.9.12}{7.10.13.16}x^4 + ...$$

Numerator Elements	**Denominator Elements**
3, 6, 9, 12 ...	7, 10, 13, 16 ...
$a = 3$	$a = 7$
$d = 6 - 3 = 3$	$d = 10 - 7 = 3$
$t_n = a + (n-1)d$	$t_n = a + (n-1)d$
$= 3 + (n-1)(3)$	$= 7 + (n-1)(3)$
$= 3n$	$= 3n + 4$

$$\frac{3}{7}x + \frac{3.6}{7.10}x^2 + ... = \sum \frac{3.6...3n}{7.10...(3n+4)} x^n$$

$$u_n = \frac{3.6...3n}{7.10...(3n+4)}x^n \quad ... (1)$$

$$u_{n+1} = \frac{3.6...3n[3(n+1)]x^{n+1}}{7.10...(3n+4)[3(n+1)+4]}$$

$$\Rightarrow \quad u_{n+1} = \frac{3.6...3n(3n+3)]x^n.x^1}{7.10...(3n+4)(3n+7)} \quad ... (2)$$

Dividing equation (2) by equation (1),

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{3.6...(3n)(3n+3)]x^n.x^1}{7.10...(3n+4)(3n+7)} \times \frac{7.10...(3n+4)}{(3.6...3n)x^n}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{(3n+3)x}{3n+7}$$

Applying limit on both sides,

$$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{n\left(3+\frac{3}{n}\right)x}{n\left(3+\frac{7}{n}\right)}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(3+\frac{3}{n}\right)x}{\left(3+\frac{7}{n}\right)}$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = x$$

By Ratio test, $\sum u_n$

(i) Converges if $x < 1$

(ii) Diverges if $x > 1$

(iii) At $x = 1$, ratio test fail.

Applying Raabe's test for $x = 1$,

$$\frac{u_{n+1}}{u_n} = \frac{(3n+3)x}{(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{(3n+3)x}$$

Substituting $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

Substituting '1' on both sides,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{3n+7}{3n+3} - 1$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{3n+7-3n-3}{3n+3}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4}{3n+3}$$

Multiplying 'n' on both sides,

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{4n}{3n+3}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{4n}{n\left[3+\frac{3}{n}\right]}$$

$$\underset{n\to\infty}{\text{Lt}}\, n\left[\frac{u_n}{u_{n+1}} - 1\right] = \underset{n\to\infty}{\text{Lt}} \frac{4}{3+\frac{3}{n}}$$

$$= \frac{4}{3+0}$$

$$= \frac{4}{3} > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is convergent for $x = 1$

$\therefore$ The given series converges if $x \le 1$ and diverges if $x > 1$.

Q49. Test for convergence $1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{2.3} + ...$

Answer :

Given series is,

$$1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2}{2.3} + ...$$

$$\text{Let, } \Sigma u_n = \sum \frac{a(a+1)(a+2)...(a+n)}{1.2.3...n(n+1)}$$

$$\Rightarrow \quad u_n = \frac{a(a+1)...(a+n)}{1.2.3...n(n+1)} \quad ... (1)$$

$$\Rightarrow \quad u_{n+1} = \frac{a(a+1)...(a+n)(a+n+1)}{1.2.3...n(n+1)(n+2)} \quad ... (2)$$

Dividing equation (2) by equation (1)

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{a(a+1)...(a+n)(a+n+1)}{1.2.3...n(n+1)(n+2)} \times \frac{1.2.3...(n+1)}{a(a+1)...(a+n)}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{a+n+1}{n+2}$$

Substracting '1' on both sides,

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} - 1 = \frac{n+2}{a+n+1} - 1$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{n+2-a-n-1}{a+n+1}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{1-a}{a+n+1}$$

Multiplying 'n' on both sides,

$$\Rightarrow n\left[\frac{u_n}{u_{n+1}}-1\right] = \frac{(1-a)n}{a+n+1}$$

Applying limit on both sides,

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ n\left[\frac{u_n}{u_{n+1}}-1\right] = \underset{n\to\infty}{\text{Lt}}\ \frac{(1-a)n}{n\left[\frac{a}{n}+1+\frac{1}{n}\right]}$$

$$= \underset{n\to\infty}{\text{Lt}}\ \frac{(1-a)}{\frac{a}{n}+1+\frac{1}{n}}$$

$$= \frac{1-a}{0+1+0}$$

$$= 1 - a$$

By using Raabe's test, $\sum u_n$

(i) Converges if $1 - a > 1$

$-a > 0$

$a < 0$

(ii) Diverges if $1 - a < 1$

$-a < 0$

$a > 0$

Q50. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1.4.7.....(3n-2)}{2.5.8.......(3n-1)}$.

Answer : Dec.-16, Q12(a)

Given series is,

$$\sum_{n=1}^{\infty} \frac{1.4.7.....(3n-2)}{2.5.8.....(3n-1)}$$

Let, $$u_n = \frac{1.4.7.....(3n-2)}{2.5.8.....(3n-1)} \quad ...(1)$$

$$u_{n+1} = \frac{1.4.7.....(3n-2)(3(n+1)-2)}{2.5.8.....(3n-1)(3(n+1)-1)}$$

$$\Rightarrow u_{n+1} = \frac{1.4.7.....(3n-2)(3n+1)}{2.5.8.....(3n-1)(3n+2)} \quad ...(2)$$

Dividing equation (1) by equation (2),

$$\frac{u_n}{u_{n+1}} = \frac{\frac{1.4.7}{2.5.8}......\frac{(3n-2)}{(3n-1)}}{\frac{1.4.7}{2.5.8}......\frac{(3n+1)}{(3n+2)}}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{3n+2}{3n+1}$$

Substracting '1' on both sides,

$$\Rightarrow \frac{u_n}{u_{n+1}}-1 = \frac{3n+2}{3n+1}-1 = \frac{3n+2-3n-1}{3n+1} = \frac{1}{3n+1}$$

$$\Rightarrow \underset{n\to\infty}{Lim}\ n\left(\frac{u_n}{u_{n+1}}-1\right) = \underset{n\to\infty}{Lim}\ \frac{n}{3n+1}$$

$$= \underset{n\to\infty}{Lim}\ n\frac{n}{n\left(3+\frac{1}{n}\right)} = \frac{1}{3} < 1$$

$\therefore$ By Raabe's test, the series $\sum u_n$ is divergent.

Q51 Text for the convergence of series, $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + ...$

Answer :

Given series is,

$$\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + ...$$

By inspection it is clear that the given series is a series of positive terms.

Consider,

$$u_n = \frac{(a+nx)^n}{n!}$$

$$u_{n+1} = \frac{(a+(n+1)x)^{n+1}}{(n+1)!}$$

Consider,

$$\frac{u_n}{u_{n+1}} = \frac{\frac{(a+nx)^n}{n!}}{\frac{[a+(n+1)x]^{n+1}}{(n+1)!}}$$

Applying limit on both sides,

$$\Rightarrow \underset{n\to\infty}{\text{Lim}}\left[\frac{u_n}{u_{n+1}}\right] = \underset{n\to\infty}{\text{Lim}}\ \frac{\left[\frac{(a+nx)^n}{n!}\right]}{\left[\frac{(a+(n+1)x)^{n+1}}{(n+1)!}\right]}$$

$$= \underset{n\to\infty}{\text{Lim}}\ \frac{(n+1)!(a+nx)^n}{n!(a+(n+1)x)^{n+1}}$$

$$= \underset{n\to\infty}{\text{Lim}}\ \frac{(n+1).n!(a+nx)^n}{n!(a+(n+1)x)^{n+1}}$$

$$= \underset{n\to\infty}{\text{Lim}}\ \frac{(n+1)(a+nx)^n}{[a+(n+1)x]^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)\left[nx\left(\frac{a}{nx}+1\right)\right]^n}{\left[(n+1)x\left(\frac{a}{(n+1)x}+1\right)\right]^{n+1}} = \lim_{n\to\infty} \frac{(n+1)n^n.x^n\left(\frac{a}{nx}+1\right)^n}{(n+1)^{n+1}.x^{n+1}\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)n^n.x^n\left(\frac{a}{nx}+1\right)^n}{(n+1)^n.(n+1).x^{n+1}\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^n.x^n\left(\frac{a}{nx}+1\right)^n}{(n+1)^n x^n.x\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^n.\frac{1}{x}.\frac{\left(\frac{a}{nx}+1\right)^n}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n\left(1+\frac{1}{n}\right)}\right)^n.\frac{1}{x}.\frac{\left(\frac{a}{nx}+1\right)^n}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right] = \lim_{n\to\infty}\left[\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}.\frac{1}{x}.\frac{\left(\frac{a}{nx}+1\right)^n}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\left[1+\left(\frac{1}{n}\right)^n\right]}\right] \times \lim_{n\to\infty}\left(\frac{1}{x}\right) \times \lim_{n\to\infty}\left[\frac{\left(\frac{a}{nx}+1\right)^n}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right]$$

$$= \frac{1}{e} \times \frac{1}{x} \times \lim_{n\to\infty}\left[\frac{\left(\frac{a}{nx}+1\right)^n}{\left(\frac{a}{(n+1)x}+1\right)}\right] \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{e} \times \frac{1}{x} \times \frac{\lim_{n\to\infty}\left(\frac{a}{nx}+1\right)^n}{\lim_{n\to\infty}\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \frac{1}{e.x} \times \frac{e^{a/x}}{e^{a/x}} \qquad \left[\because \lim_{n\to\infty}\left(\frac{a}{nx}+1\right)^n = e^{a/x}\right]$$

$$= \frac{1}{e.x}$$

By ratio test, the given series is convergent if $\frac{1}{e.x} > 1$ i.e., $x < \frac{1}{e}$ and the series is divergent if $\frac{1}{e.x} < 1$ i.e., $x > \frac{1}{e}$.

If $x = \frac{1}{e}$ then test fails.

When $x = \frac{1}{e}$,

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{1}{\left(\frac{1}{e}\right)} \times \frac{\left(\frac{a}{n\left(\frac{1}{e}\right)}+1\right)^n}{\left(\frac{a}{(n+1)\left(\frac{1}{e}\right)}+1\right)^{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n} \times \frac{\left(\frac{ae}{n}+1\right)^n}{\left(\frac{ae}{(n+1)}+1\right)^{n+1}}$$

Since $\frac{u_n}{u_{n+1}}$ involves 'e', we apply logarithmic test.

Applying log on both sides,

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log\left[\frac{e.\left(\frac{ae}{n}+1\right)^n}{\left(1+\frac{1}{n}\right)^n\left(\frac{ae}{(n+1)}+1\right)^{n+1}}\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log e + \log\left(\frac{ae}{n}+1\right)^n - \log\left(1+\frac{1}{n}\right)^n - \log\left(\frac{ae}{(n+1)}+1\right)^{n+1}$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = 1 + n\log\left(\frac{ae}{n}+1\right) - n\log\left(1+\frac{1}{n}\right) - (n+1)\log\left(\frac{ae}{(n+1)}+1\right)$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = 1 + n\left(\frac{ae}{n} - \frac{a^2e^2}{2n^2} + \frac{a^3e^3}{3n^3}...\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}...\right) - (n+1)\left(\frac{ae}{(n+1)} - \frac{a^2e^2}{2(n+1)^2} + \frac{a^3e^3}{3(n+1)^3}...\right)$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = 1 + ae - \frac{a^2e^2}{2n} + \frac{a^3e^3}{3n^2} - ... -1 + \frac{1}{2n} - \frac{1}{3n^2} + ... -ae + \frac{a^2e^2}{2(n+1)} - \frac{a^3e^3}{3(n+1)^2} + ...$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-a^2e^2}{2n} + \frac{a^3e^3}{3n^2} + \frac{1}{2n} - \frac{1}{3n^2} + \frac{a^2e^2}{2(n+1)} - \frac{a^3e^3}{3(n+1)^2} + ...$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1-a^2e^2}{2n} + \frac{a^2e^2}{2(n+1)} + ...$$

Multiplying 'n' on both sides,

$$\Rightarrow \quad n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1-a^2e^2}{2} + \frac{n.a^2e^2}{2(n+1)} + ...$$

Applying limit on both sides,

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \lim_{n\to\infty}\left[\frac{1-a^2e^2}{2} + \frac{na^2e^2}{2(n+1)} + \ldots\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \lim_{n\to\infty}\frac{1-a^2e^2}{2} + \lim_{n\to\infty}\frac{na^2e^2}{2(n+1)} + \ldots$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2}(1) + \lim_{n\to\infty}\frac{na^2e^2}{2n\left(1+\frac{1}{n}\right)} + \ldots \qquad \left[\because \lim_{n\to\infty}(1) = 1\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2} + \lim_{n\to\infty}\frac{a^2e^2}{2\left(1+\frac{1}{n}\right)} + \ldots$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2} + \frac{a^2e^2}{2\left(1+\frac{1}{\infty}\right)} + \ldots$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2} + \frac{a^2e^2}{2(1+0)} + \ldots$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \left(\frac{1}{2} - \frac{a^2e^2}{2}\right) + \frac{a^2e^2}{2(1)} + \ldots$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1}{2} - \frac{a^2e^2}{2} + \frac{a^2e^2}{2} + \ldots$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1}{2} < 1$$

Since, the value obtained is less than one, by logarithmic test the given series is divergent.

Thus the series converges if $x < \frac{1}{e}$ and diverges if $x \geq \frac{1}{e}$.

Q52. Test the following series for the convergence,

$$\mathbf{1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \ldots}$$

Answer :

Given series is,

$$1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \ldots$$

From the given series,

$$u_n = \frac{n!x^n}{(n+1)^n}$$

$$u_{n+1} = \frac{(n+1)!x^{n+1}}{(n+2)^{n+1}}$$

Consider,

$$\frac{u_n}{u_{n+1}} = \frac{\left[\dfrac{n!x^n}{(n+1)^n}\right]}{\left[\dfrac{(n+1)!x^{n+1}}{(n+2)^{n+1}}\right]}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{n!.x^n.(n+2)^{n+1}}{(n+1)^n.(n+1)!x^{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{n!.x^n.(n+2)^{n+1}}{(n+1)^n.(n+1)n!x^n.x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(n+2)^{n+1}}{(n+1).(n+1)^n.x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\frac{(n+2)}{(n+1)}\right]^{n+1}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\frac{(n+1)+1}{(n+1)}\right]^{n+1}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\left(\frac{n+1}{n+1}+\frac{1}{(n+1)}\right)^{n+1}\right].\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\left(1+\frac{1}{n+1}\right)^{n+1}\right].\frac{1}{x}$$

Applying limit on both sides,

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left\{\left[1+\frac{1}{(n+1)}\right]^{n+1}.\frac{1}{x}\right\}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{x}\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{x}.e \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{e}{x}$$

By ratio test, it is clear that the series is convergent if $\frac{e}{x} > 1$ i.e., $x < e$ and diverges if $\frac{e}{x} < 1$ i.e., $x > e$ and the test fails when $\frac{e}{x} = 1 \Rightarrow x = e$.

When $x = e$,

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n+1}\right)^{n+1}.\frac{1}{e}$$

since $\frac{u_n}{u_{n+1}}$ involves e we apply logarithmic test. Applying log on both sides,

$$\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left[\frac{1}{e}\left(1+\frac{1}{(n+1)}\right)^{n+1}\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log\frac{1}{e} + \log\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log e^{-1} + (n+1)\log\left(1+\frac{1}{n+1}\right)$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = -\log e + (n+1)\left(\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots\right) \quad \left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = -1 + (n+1)\left[\frac{1}{(n+1)} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = -1 + 1 - \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} + \ldots$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-1}{2(n+1)} + \frac{1}{3(n+1)^2} + \ldots$$

Multiply 'n' on both sides,

$$\Rightarrow n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-n}{2(n+1)} + \frac{n}{3(n+1)^2} + \ldots$$

$$\Rightarrow n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-n}{2n\left(1+\frac{1}{n}\right)} + \frac{n}{3n^2\left(1+\frac{1}{n}\right)^2} + \ldots$$

$$\Rightarrow \quad n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-1}{2\left(1+\frac{1}{n}\right)} + \frac{1}{3n\left(1+\frac{1}{n}\right)^2}$$

Applying limit on both sides,

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \underset{n\to\infty}{Lt}\left[\frac{-1}{2\left(1+\frac{1}{n}\right)} + \frac{1}{3n\left(1+\frac{1}{n}\right)^2} + \ldots\right]$$

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{-1}{2\left(1+\frac{1}{\infty}\right)} + \frac{1}{3\infty\left(1+\frac{1}{\infty}\right)^2} + \ldots$$

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{-1}{2(1+0)} + 0...$$

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{-1}{2(1)} = \frac{-1}{2} < 1$$

$\therefore$ The series is divergent series.

Thus, the series converges if $x < e$ and diverges if $x \geq e$.

1.4 ALTERNATING SERIES, SERIES OF POSITIVE AND NEGATIVE TERMS

Q53. State and prove Leibnitz's test.

Answer :

If $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is an alternating series then it is convergent if,

(a) $u_1 \geq u_2 > u_3 \geq ... \geq u_n \geq u_{n+1}$ ....

(b) $\underset{n\to\infty}{Lt}\, u_n = 0$

Proof

Consider $s_n = u_1 - u_2 + u_3 - u_4 + ... (-1)^{n-1} u_n$

And, $s_{2n} = u_1 - u_2 + u_3 - u_4 + ... + u_{2n-1} - u_{2n}$ ... (1)

$s_{2n+2} = u_1 - u_2 + u_3 - u_4 + ...$ ... (2)

$u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$ ... (3)

By substracting equation (3) from equation (2),

$s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geq 0$ [$\because$ From condition (a)]

$\therefore$ $s_{2n+2} \geq s_{2n} \ \forall\, n$

The subsequence $\{s_{2n}\}$ of $\{s_n\}$ is an increasing sequence.

From equation (2),

$s_{2n} = u_1 - [(u_2 - u_3) + ... + (u_{2n-2} - u_{2n-1}) + u_{2n}]$

$= u_1$ – a positive number [$\because$ $u_2 - u_3, ... u_{2n-2} - u_{2n-1}, u_{2n}$ are positive]

$\therefore$ $s_{2n} \leq u_1 \ \forall\, n$

$\therefore$ $\{s_{2n}\}$ is bounded above

$\therefore$ $\{s_{2n}\}$ converges.

Assume $\lim_{n\to\infty} s_{2n = l}$ ... (4)

$\because$ $s_{2n-1} = s_{2n} + u_{2n}$

$\therefore$ $\lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} u_{2n}$

$= l + 0$

[$\because$ From equation (4) and condition (b)]

$\therefore$ $\{s_{2n-1}\}$ also converges to l

$\therefore$ $\{s_n\}$ converges to l

Hence, an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ also convergent.

Q54. Discuss the convergence of the series $\sum \frac{\cos n\pi}{n^2+1}$.

Answer :

Given that,

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$= \frac{-1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + ... + \frac{(-1)^n}{n^2+1}$$

$$u_1 = \frac{1}{2}, u_2 = \frac{1}{5}, u_3 = \frac{1}{10}, u_4 = \frac{1}{17}, ..., u_n = \frac{1}{n^2+1}$$

$u_1 > u_2 > u_3 > ... u_{n-1} > u_n$

$\Rightarrow \quad u_{n-1} > u_n$

$\Rightarrow \quad u_n < u_{n-1}$

And $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt} \frac{1}{n^2+1} = 0$

$\therefore$ From Leibnitz's test the given series is convergent.

$\therefore \quad \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$ is convergent.

Q55. Discuss the convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + ... \ (0 < x < 1)$$

Answer :

Given that,

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} ...... = \sum \frac{x^n}{1+x^n}(-1)^{n-1}$$

Let, $u_n = \frac{x^n}{1+x^n}$

Since,

$1 + x^{n-1} < 1 + x^n$

$\Rightarrow \quad \frac{1}{1+x^{n-1}} > \frac{1}{1+x^n}$

$\Rightarrow \quad \frac{x^{n-1}}{1+x^{n-1}} > \frac{x^n}{1+x^n}$

$\Rightarrow \quad u_{n-1} > u_n$

$\Rightarrow \quad u_n < u_{n-1}$

$$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{x^n}{1+x^n} = \underset{n\to\infty}{\text{Lt}} \frac{x^n}{x^n\left(\frac{1}{x^n}+1\right)} = \frac{1}{\infty+1} = 0$$

From Leibnitz test the given series is convergent.

$\therefore \quad \sum \frac{x^n}{1+x^n}(-1)^{n-1}$ is convergent.

Q56. Examine the convergence of the alternating series $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}$ ...

OR

By Leibnitz's test, verify the series $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\frac{1}{7.8}$ ... is convergent.

Answer :

Given series is,

$$\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-...$$

First Elements

1, 3, 5 ...

$a = 1,$

$d = 3 - 1 = 2$

$t_n = a + (n - 1)\, d$

$= 1 + (n - 1)\,(2)$

$= 2n - 1$

Second Elements

2, 4, 6, 8 ...

$a = 4$

$d = 4 - 2 = 2$

$t_n = a + (n - 1)\, d = 2 + (n - 1)\,(2) = 2n$

$$\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}\ ...\ = \Sigma\,(-1)^{n-1}\,\frac{1}{(2n-1)\,(2n)}$$

Let,

$$u_n = \frac{1}{(2n-1)\,(2n)}$$

$$u_{n-1} = \frac{1}{(2n-3)\,(2n-2)}$$

Consider,

$$u_n - u_{n-1} = \frac{1}{(2n-1)\,(2n)} - \frac{1}{(2n-3)\,(2n-2)}$$

$$= \frac{(2n-3)\,(2n-2)-(2n-1)\,(2n)}{(2n)\,(2n-1)\,(2n-2)\,(2n-3)}$$

$$< 0$$

$\therefore \quad u_n - u_{n-1} < 0$

$\therefore \quad u_n < u_{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n-1)\,(2n)} = \frac{1}{\infty} = 0$$

From Leibnitz test $\Sigma\,(-1)^{n-1}\,\frac{1}{(2n-1)\,(2n)}$ is convergent.

Q57. State Leibnitz test and use it to test the convergence of the series $\sum(-1)^n\frac{n}{2n+1}$.

Answer :

Leibnitz Test

For answer refer Unit-1, Q14.

Given series is,

$$\sum(-1)^n\frac{n}{2n+1}$$

The terms of the series are alternatively positive and negative.

Let,

$$u_n = \frac{n}{2n+1}$$

$$\Rightarrow \quad u_{n+1} = \frac{n}{2(n+1)+1} = \frac{n+1}{2n+3}$$

$$u_n - u_{n+1} = \frac{n}{2n+1} - \frac{n+1}{2n+3}$$

$$= \frac{2n^2+3n-2n^2-2n-n-1}{(2n+1)(2n+3)}$$

$$= \frac{-1}{(2n+1)(2n+3)}$$

$\therefore \quad u_n < u_{n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n}{n(2+1/n)}$$

$$= \lim_{n\to\infty} \frac{1}{2+1/n} = \frac{1}{2} \neq 0$$

Both conditions of convergence are not satisfied.

$\therefore$ By Leibnitz test, the series is not convergent. It is oscillatory.

1.5 ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

Q58. Test the series $\Sigma(-1)^{n-1}\frac{1}{n2^n}$ for absolute convergence.

Answer :

Given series,

$$\Sigma(-1)^{n-1}\frac{1}{n2^n}$$

Let, $u_n = (-1)^{n-1}\frac{1}{n2^n}$

$$\Rightarrow \quad u_{n+1} = (-1)^{n+1-1}\frac{1}{(n+1)2^{n+1}} = \frac{(-1)^n}{(n+1)2^{n+1}}$$

Consider,

$$\frac{u_{n+1}}{u_n} = \frac{\frac{(-1)^n}{(n+1)2^{n+1}}}{\frac{(-1)^{n-1}}{n.2^n}}$$

$$= \frac{(-1)^n}{(n+1)2^{n+1}} \times \frac{n.2^n}{(-1)^{n-1}}$$

$$= \frac{(-1)^n}{(n+1).2^n.2^1} \times \frac{n.2^n}{(-1)^n.(-1)^{-1}}$$

$$= \frac{n}{2(n+1).(-1)^{-1}} = \frac{n(-1)}{2(n+1)}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{-n}{2(n+1)} \qquad \left[\because \frac{1}{(-1)^{-1}} = -1\right]$$

Then, $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{-n}{2(n+1)}\right|$

$$\Rightarrow \quad \mathop{Lt}_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \mathop{Lt}_{n\to\infty}\frac{n}{2(n+1)}$$

$$= \mathop{Lt}_{n\to\infty}\frac{n}{2n\left(1+\frac{1}{n}\right)}$$

$$= \mathop{Lt}_{n\to\infty}\frac{1}{2\left(1+\frac{1}{n}\right)} = \frac{1}{2\left(1+\frac{1}{\infty}\right)}$$

$$= \frac{1}{2(1+0)} = \frac{1}{2} < 1$$

$$\Rightarrow \quad \mathop{Lt}_{n\to\infty}\frac{|u_{(n+1)}|}{|u_n|} < 1$$

$\therefore \quad \Sigma u_n$ is absolute convergent.

Q59. Test the following series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ for conditional convergence.

Answer : Dec.-17, Q12(b)

Given series is,

$$\sum(-1)^{n-1}\frac{n}{n^2+1}$$

Let,

$$u_n = (-1)^{n-1}.\frac{n}{n^2+1}$$

$$\Rightarrow \quad u_n = (-1)^{n-1}.\, v_n$$

Where,

$$v_n = \frac{n}{n^2+1}$$

Σu_n is an a alternating series.

i.e., It contains terms which are alternatively positive and negative.

Consider,

$$v_n = \frac{n}{n^2+1}$$

$$\Rightarrow \quad v_n > 0 \; \forall \; n$$

$$v_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$\therefore \quad v_n > v_{n+1} \ \forall \ n$$

Also, $\underset{n\to\infty}{Lt}\ v_n = \underset{n\to\infty}{Lt}\ \frac{n}{n^2+1}$

$$= \underset{n\to\infty}{Lt}\ \frac{n}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \underset{n\to\infty}{Lt}\ \frac{1}{n}\left(1+\frac{1}{n^2}\right)$$

$$= 0$$

Since, $v_n > 0$, $v_n > v_{n+1}$ and $\underset{n\to\infty}{Lt}\ v_n = 0$

Thus, by Leibnitz's test the alternating series is a convergent series.

$\therefore$ Σu_n is convergent

Test for Absolute Convergence or Conditional Convergence

$$u_n = (-1)^{n-1} . \frac{n}{n^2+1}$$

$$\Rightarrow \quad |u_n| = \left|(-1)^{n-1} . \frac{n}{n^2+1}\right|$$

$$\Rightarrow \quad |u_n| = \frac{n}{n^2+1}$$

Let,

$$\Sigma v_n = \sum \frac{1}{n}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\ \frac{|u_n|}{v_n} = \underset{n\to\infty}{Lt}\ \frac{n}{n^2+1} . \frac{1}{\frac{1}{n}} = \underset{n\to\infty}{Lt}\ \frac{n^2}{n^2+1}$$

$$= \underset{n\to\infty}{Lt}\ \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = \underset{n\to\infty}{Lt}\ \frac{1}{1+\frac{1}{n^2}}$$

$$= 1 \neq 0$$

$\therefore$ By comparison test, Σv_n is divergent

Hence, $\Sigma|u_n|$ is also divergent.

$\therefore$ Σv_n is convergent but not absolutely convergent i.e., it converges conditionally.

Q60. Prove that the series $\sum (-1)^{n-1} \frac{\sin nx}{n^2}$ converges absolutely.

Answer : June/July-17, Q12(b)

Given series is,

$$\Sigma(-1)^{n-1} \frac{\sin nx}{n^2}$$

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\therefore\ \Sigma(-1)^{n-1}\dfrac{\sin nx}{n^2}$ is convergent if $\Sigma\left|(-1)^{n-1}\dfrac{\sin nx}{n^2}\right|$ converges.

$$\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2} \qquad [\because |\sin nx| \leq 1]$$

$\Sigma\dfrac{1}{n^2}$ is a power series with power = 2 which is greater than 1.

$\therefore\ \Sigma\dfrac{1}{n^2}$ is convergent, and hence $\Sigma(-1)^{n-1}\dfrac{\sin nx}{n^2}$ is absolutely convergent.

Q61. Examine whether the series

$-1+\dfrac{1}{2^2}-\dfrac{1}{3^2}+\dfrac{1}{4^2}-\dfrac{1}{5^2}+....$ is absolutely convergent or conditionally convergent.

Answer : Dec.-16, Q12(b)

Given series is,

$$-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+.....$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+.....$$

Let,

$$\sum_{n=1}^{\infty}u_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty}(-1)\,v_n$$

Where, $v_n = \dfrac{1}{n^2} > 0\ \forall n$

$$v_{n+1} = \frac{1}{(n+1)^2}$$

Since $n^2 < (n+1)^2$;

$\Rightarrow\ v_n > v_{n+1}\ \forall n$

$\Rightarrow\ \underset{n\to\infty}{lt}\ v_n = \underset{n\to\infty}{lt}\ \dfrac{1}{n^2} = 0$

$\therefore$ By Leibnitz's test, the given series is convergent

As $u_n = (-1)^n\dfrac{1}{n^2}$;

$|u_n| = \dfrac{1}{n^2}$;

By p-test, $|u_n| = \dfrac{1}{n^2}$ is convergent with $p = 2$.

As $|u_n|$ is convergent, Σu_n is absolutely convergent.

Hence, the given series $-1+\dfrac{1}{2^2}-\dfrac{1}{3^2}+\dfrac{1}{4^2}-\dfrac{1}{5^2}+....$ is absolutely convergent.

UNIT 2

CALCULUS OF ONE VARIABLE

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. State Rolle's theorem.

Answer :

If $\phi(x)$ is any function in the closed interval $[a, b]$ such that,

(i) $\phi(x)$ is continuous in $[a, b]$

(ii) $\phi(x)$ is differentiable in (a, b)

(iii) $\phi(a) = \phi(b)$

Then there exists at least one point $x = c$ in (a, b) such that $a < c < b$ and $\phi'(c) = 0$.

Q2. Verify Rolle's theorem for $f(x) = x^2$ in [–1, 1].

Answer :

Given function is,

$f(x) = x^2$ in $[-1, 1]$ ... (1)

(i) To Check Continuity of f(x)

$f(x)$ is a polynomial in x, therefore $f(x)$ is continuous in closed interval $[-1, 1]$.

(ii) To Check Differentiability of f(x)

$f(x)$ is differentiable in open interval $(-1, 1)$. Since $f'(x) = 2x$ is defined in $(-1, 1)$.

(iii) To Check f(a) = f(b)

$f(-1) = (-1)^2$

$= 1$

$f(1) = (1)^2$

$= 1$

$\therefore$ $f(-1) = f(1)$

Hence, $f(x)$ satisfies all the three conditions of Rolle's theorem.

A point c exists and $c \in [-1, 1]$ such that $f'(c) = 0$

$\Rightarrow$ $f'(c) = 2c = 0$

$\Rightarrow$ $2c = 0$

$\therefore$ $c = 0$ lies in the range $[-1, 1]$

Hence, Rolle's theorem is verified.

Q3. Find the c value of Rolle's mean value theorem for the function $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ on [a, b].

Answer :

Given function is,

$$f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right) \text{ in } [a, b]$$

(i) To check continuity of $f(x)$

$f(x)$ is a composite function of continuous functions in $[a, b]$, therefore $f(x)$ is continuous in closed interval $[a, b]$.

(ii) To check differentiability of $f(x)$

$$f'(x) = \frac{1}{a+b}\left[\frac{x(2x)-(x^2+ab)}{x^2}\right]\frac{x(x+b)}{x^2+ab}$$

$$= \frac{x^2-ab}{x(x^2+ab)}$$

$\therefore$ $f(x)$ is differentiable in open interval (a, b).

(iii) To check $f(a) = f(b)$

$$f(a) = \log\left(\frac{a^2+ab}{a(a+b)}\right)$$

$$= \log\left(\frac{a^2+ab}{a^2+ab}\right)$$

$$= \log 1$$

$$= 0$$

$$f(b) = \log\left(\frac{b^2+ab}{b(a+b)}\right)$$

$$= \log\left(\frac{b^2+ab}{b^2+ab}\right)$$

$$= \log 1$$

$$= 0$$

$\therefore$ $f(a) = f(b)$

Hence, $f(x)$ satisfies all the conditions of Rolle's theorem. A point $c \in (a, b)$ exits such that $f'(c) = 0$

$\Rightarrow$ $\frac{c^2-ab}{c(c^2+ab)} = 0$

$\Rightarrow$ $c^2 - ab = 0$

$\Rightarrow$ $c^2 = ab$

$\Rightarrow$ $c = \pm\sqrt{ab}$

$\therefore$ $c = \pm\sqrt{ab} \in (a,b)$

Q4. State Lagrange's mean value theorem.

Answer :

If $f(x)$ is a function defined in $[a, b]$ such that,

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is derivable in (a, b)

Then, there exist atleast one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Q5. Show that $\frac{2}{\pi} < \frac{\sin x}{x} < 1, 0 < x < \frac{\pi}{2}$.

Answer :

Let, $f(x) = \frac{\sin x}{x}$ ... (1)

Since, $\lim_{x \to 0} f(x) = 1$

Then $f(0) = 1$ ... (2)

Differentiating equation (1) with respect to 'x',

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

Since for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ $x < \tan x$

$\Rightarrow$ $x < \frac{\sin x}{\cos x}$

$\Rightarrow$ $x \cos x < \sin x$

$\therefore$ $f'(x) < 0$

From Mean value theorem,

If $x \in \left(0, \frac{\pi}{2}\right)$ then $\frac{f(x)-f(0)}{x} = f'(c_1) < 0$

$\Rightarrow$ $\frac{f(x)-f(0)}{x} < 0$

$\Rightarrow$ $f(x) - f(0) < 0$

$\Rightarrow$ $f(x) < f(0)$

$\Rightarrow$ $\frac{\sin x}{x} < 1$ [$\because$ From equations (1) and (2)] ... (3)

Consider,

$$\frac{f\left(\frac{\pi}{2}\right) - f(x)}{\left(\frac{\pi}{2}\right) - x} = f'(c_2) < 0$$

$\Rightarrow$ $\frac{f\left(\frac{\pi}{2}\right) - f(x)}{\frac{\pi}{2} - x} < 0$

$\Rightarrow$ $f\left(\frac{\pi}{2}\right) - f(x) < 0$

$\Rightarrow$ $f\left(\frac{\pi}{2}\right) < f(x)$

$\Rightarrow$ $\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} < \frac{\sin x}{x}$

$\Rightarrow$ $\frac{1}{\frac{\pi}{2}} < \frac{\sin x}{x}$

$\Rightarrow$ $\frac{2}{\pi} < \frac{\sin x}{x}$ ... (4)

$\therefore$ From equations (3) and (4),

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1$$

Q6. Show that between any two roots of e^x cos x = 1, there exists at least one root of e^x sin x = 1.

Answer :

Given function is,

$e^x \cos x = 1$ ... (1)

Let, $f(x) = e^x \cos x - 1$

Let the two roots of equation (1) be a and b

Then,

$e^a.\cos a = 1$; $e^b.\cos b = 1$

$\Rightarrow \cos a = \dfrac{1}{e^a}$; $\cos b = \dfrac{1}{e^b}$

$\Rightarrow \cos a = e^{-a}$; $\cos b = e^{-b}$

$\Rightarrow e^{-a} - \cos a = 0$ .... (2) ; $e^{-b} - \cos b = 0$ ... (3)

From equations (2) and (3),

$f(x)$ can be rewritten as,

$f(x) = e^{-x} - \cos x$

$\therefore$ $f(x) = e^{-x} - \cos x$ is continous on $[a, b]$ and differentiable on (a, b)

$\therefore$ $f(a) = e^{-a} - \cos a = 0$

$f(b) = e^{-b} - \cos b = 0$

Hence, $f(x)$ satisfies the conditions of Rolle's theorem,

$\therefore$ A point $c \in (a, b)$ exists such that $f'(c) = 0$

$f'(x) = -e^{-x} - (-\sin x) = 0$

$f'(c) = 0 \Rightarrow -e^{-c} - (-\sin c) = 0$

$\Rightarrow -e^{-c} + \sin c = 0$

$\Rightarrow \sin c = e^{-c}$

$\Rightarrow \sin c = 1/e^c$

$\Rightarrow e^c \sin c = 1$

$\therefore$ $e^c \sin c - 1 = 0$; $c \in (a, b)$

Q7. State Cauchy's mean value theorem.

Answer :

If $f : [a, b] \to R$, $g : [a, b] \to R$ are such that,

(i) f and g are continuous on $[a, b]$

(ii) f and g are differentiable on (a, b) and

(iii) $g'(x) \neq 0 \ \forall x \in (a, b)$

Then, there exists a point $c \in (a, b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Q8. Prove using mean value theorem, |sinu – sinv| ≤ |u – v|.

Answer :

Let, $f(x) = \sin x$ and $g(x) = x$

Among the two limits u and v because of the unawareness of which one is greater we consider modulus.

According to Cauchy's mean value theorem,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{|\sin b - \sin a|}{|b - a|} = \frac{\cos c}{1}$$

$$\Rightarrow \frac{|\sin b - \sin a|}{|b - a|} = \cos c \quad \text{... (1)}$$

Maximum value of cos x is 1 therefore maximum value of cos(c) can be lesser than or equal to 1.

Hence, $\cos c \leq 1$

From equation (1),

$$\frac{|\sin b - \sin a|}{|b - a|} \leq 1$$

$\Rightarrow$ $|\sin b - \sin a| \leq |b - a|$

$\therefore$ $|\sin u - \sin v| \leq |u - v|$

Q9. State Taylor's theorem.

Answer :

When $f(x)$, $f'(x)$, $f''(x)$ derivatives are continuous in $[a, a + h]$ and the n^{th} derivative of x exists in $(a, a + h)$, then a number θ lies between 0 and 1, such that the following series represents Taylor's theorem with Lagrange's form remainder.

$$f(a + h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots \frac{h^n}{n!} f^n(a + \theta h)$$

Where,

$$\frac{h^n}{n!} f^n(a + \theta h) = R_n \text{ (Remainder)}$$

Q10. Find the Taylor series of f(x) = sin x about x = π/4.

Answer : June-13, Q5

Given that,

$f(x) = \sin x$, $x = \pi/4$

The Taylor's series expansion for function $f(x)$ is,

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \quad \text{... (1)}$$

Here,

$f(x) = \sin x$

$a = \pi/4$

$$\Rightarrow f'(x) = \frac{d}{dx}[f(x)]$$

$$= \frac{d}{dx}[\sin x]$$

$f'(x) = \cos x$

Similarly,

$$f''(x) = \frac{d}{dx}[f'(x)]$$

$$= \frac{d}{dx}[\cos x]$$

$$f''(x) = -\sin x$$

Also,

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$$

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$

$$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$$

$$\therefore \quad f''\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

On substituting the values of $f\left(\frac{\pi}{4}\right)$, $f'\left(\frac{\pi}{4}\right)$ and $f''\left(\frac{\pi}{4}\right)$ in equation (1), we get,

$$\sin x = \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right)}{1!} + \frac{\left(x-\frac{\pi}{4}\right)^2\left(\frac{-1}{\sqrt{2}}\right)}{2!} + \ldots$$

$$\therefore \quad \sin x = \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)}{\sqrt{2}} - \frac{\left(x-\frac{\pi}{4}\right)^2}{2\sqrt{2}} + \ldots$$

Is the requirement Taylor's series expansion.

Q11. Find the Taylor's series expansion of f(x) = 2^x about x = 0.

Answer : Dec.-13, Q1

The given function is,

$f(x) = 2^x; x = 0$

From Taylor's series expansion, we have,

$$f(x)=f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots \quad \ldots (1)$$

Here,

$f(x)= 2^x$ and $a = 0$

$\Rightarrow \quad f(a)= f(0) = 2^0 = 1$

On differentiating $f(x)$, we get,

$\Rightarrow \quad f'(x) = 2^x \log_e 2, f'(0) = 2^0 \log_e 2 = 0.6931 \quad \left[\because \frac{d}{dx}a^x = a^x \log_e a\right]$

$\Rightarrow \quad f''(x) = 2^x (\log_e 2)^2, f''(0) = 2^0 (\log_e 2)^2 = 0.4804$

$\Rightarrow \quad f'''(x) = 2^x (\log_e 2)^3, f'''(0) = 2^0 (\log_e 2)^3 = 0.3330$

Substituting the corresponding values in equation (1),

$$\Rightarrow \quad f(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f'''(0) + ...$$

$$= 1 + x(0.6931) + \frac{x^2}{2!}(0.4804) + \frac{x^3}{3!}(0.3330) + ...$$

$$= 1 + 0.6931x + \frac{x^2(0.4804)}{2} + \frac{x^3(0.3330)}{6} + ...$$

$$f(x) = 1 + 0.6931x + 0.2402x^2 + 0.0555x^3 + ...$$

$$\therefore f(x) = 2^x = 1 + 0.6931x + 0.2402x^2 + 0.0555x^3 + ...$$

Q12. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point P(1, 1).

Answer : **May/June-18, Q6**

Given curve is,

$$x^4 + y^4 = 2$$

$$\Rightarrow \quad y^4 = 2 - x^4 \qquad ... (1)$$

The radius of curvature for a curve at a point P in Cartesian form $y = f(x)$ is given as,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)_P^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)_P} \qquad ... (2)$$

Differentiating equation (1) with respect to 'x',

$$4y^3 \frac{dy}{dx} = -4x^3$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x^3}{y^3} = -\left(\frac{x}{y}\right)^3 \qquad ... (3)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = -\left(\frac{1}{1}\right)^3$$

$$= -1$$

Differentiating equation (3) with respect to 'x',

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\left(\frac{x}{y}\right)^3\right]$$

$$= -\left[\frac{y^3(3x^2) - x^3(3y^2)\left(\frac{dy}{dx}\right)}{(y^3)^2}\right]$$

$$= -\left[\frac{3x^2y^3 - 3y^2x^3\left(-\left(\frac{x}{y}\right)^3\right)}{y^6}\right]$$

$$= -\left[\frac{3x^2y^3 + 3x^3y^2\left(\frac{(x^3)}{y^3}\right)}{y^6}\right]$$

$$= -\left[\frac{3x^2y^4 + 3x^6}{y^7}\right]$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\left[\frac{3x^2y^4 + 3x^6}{y^7}\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1,1)} = -\left[\frac{3(1)^2(1)^4+3(1)^6}{(1)^7}\right] = -\left[\frac{3+3}{1}\right] = -6$$

Substituting the corresponding values in equation (2),

$$\rho_{(1,1)} = \frac{\left[1+(-1)^2\right]^{\frac{3}{2}}}{-6} = \frac{(2)^{\frac{3}{2}}}{-6} = -0.47$$

$\therefore$ The radius of curvature of the curve $x^4 + y^4 = 2$ is -0.47.

Q13. Find the radius of curvature at the origin of the curve $x^4 - 4x^3 - 18x^2 - y = 0$.

Answer : June/July-17, Q5

Given curve is,

$$x^4 - 4x^3 - 18x^2 - y = 0$$

$$\Rightarrow \quad y = x^4 - 4x^3 - 18x^2 \qquad \text{... (1)}$$

Dividing equation (1) on both sides by y,

$$1 = \frac{x^4}{y} - \frac{4x^3}{y} - \frac{18x^2}{y}$$

Applying limit as $x \to 0$ to above equation,

$$1 = \underset{x\to 0}{Lt}\left[\frac{x^4}{y} - \frac{4x^3}{y} - \frac{18x^2}{y}\right]$$

$$\Rightarrow \quad 1 = \underset{x\to 0}{Lt}\left[\frac{x^4}{y}\right] - \underset{x\to 0}{Lt}\left[\frac{4x^3}{y}\right] - 18\underset{x\to 0}{Lt}\left[\frac{x^2}{y}\right]$$

$$\Rightarrow \quad 1 = 0 - 0 - 18\left[2\rho_{at(0,0)}\right] \qquad \left[\because \lim_{x\to 0}\frac{x^2}{y} = 2\rho_{at(0,0)}\right]$$

$$\therefore \quad \rho_{at(0,0)} = \frac{-1}{36}.$$

Q14. Find the radius of curvature at the origin for the curve $x^4 - y^4 + x^3 - y^3 + x^2 - y^2 + y = 0$.

Answer :

Given curve is,

$$x^4 - y^4 + x^3 - y^3 + x^2 - y^2 + y = 0 \qquad \text{... (1)}$$

The radius of curvature for a curve at origin is given as,

$$\rho_0 = \frac{(1+a^2)^{\frac{3}{2}}}{b} \qquad \text{... (2)}$$

Let,

$$y = ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \ldots \qquad \text{... (3)}$$

Substituting equation (3) in equation (1),

$$x^4 - \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \ldots\right)^4 + x^3 - \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \ldots\right)^3 + x^2 - \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \ldots\right)^2 + \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \ldots\right) = 0$$

Comparing coefficients of 'x' on both sides,

$$a = 0$$

Comparing coefficients of 'x^2' on both sides,

$$1 - a^2 + \frac{b}{2} = 0.$$

$$\Rightarrow \quad 1 - 0 + \frac{b}{2} = 0$$

$$\Rightarrow \quad 2 + b = 0$$

$$\therefore \quad b = -2$$

Substituting the corresponding values in equation (2),

$$\rho_0 = \frac{(1+0)^{\frac{3}{2}}}{-2}$$

$$\Rightarrow \quad \rho_0 = \frac{(1)^{\frac{3}{2}}}{-2}$$

$$\therefore \quad \rho_0 = \frac{-1}{2}$$

Q15. Find the radius of curvature of the curve x = a cost t, y = b sin t at t = $\frac{\pi}{4}$.

Answer :

Given curves are,

$$x = a\cos t \qquad \text{... (1)}$$

$$y = b\sin t \qquad \text{... (2)}$$

The radius of curvature for curve in parametric form is given as,

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\left[\frac{d^2y}{dt^2}\right] - \frac{dy}{dt}\left[\frac{d^2x}{dt^2}\right]} \qquad \text{... (3)}$$

Differentiating equation (1), 2 times with respect to 't',

$$\frac{dx}{dt} = a[-\sin t] = -a\sin t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = \frac{d}{dt}[-a\sin t] = -a\cos t$$

Differentiating equation (2), 2 times with respect to t,

$$\frac{dy}{dt} = \frac{d}{dt}(b\sin t)$$

$$\Rightarrow \quad \frac{dy}{dt} = b(\cos t)$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}[b\cos t] = b[-\sin t]$$

Substituting the corresponding values in equation (3),

$$\rho = \frac{[a^2\sin^2 t + b^2\cos^2 t]^{\frac{3}{2}}}{-a\sin t[-b\sin t] - b\cos t[-a\cos t]}$$

$$= \frac{[a^2\sin^2 t + b^2\cos^2 t]^{\frac{3}{2}}}{ab(\sin^2 t + \cos^2 t)}$$

$$= \frac{[a^2\sin^2 t + b^2\cos^2 t]^{\frac{3}{2}}}{ab}$$

$$\rho \text{ at } t = \frac{\pi}{4} = \frac{\left[a^2\left(\frac{1}{\sqrt{2}}\right)^2 + b^2\left(\frac{1}{\sqrt{2}}\right)^2\right]^{\frac{3}{2}}}{ab} = \frac{[a^2+b^2]^{\frac{3}{2}}}{2\sqrt{2}\,ab}$$

Q16. Show that the curvature of the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the Folium $x^3 + y^3 = 3axy$ is $\frac{-8\sqrt{2}}{3a}$.

Answer :

Given curve is,

$$x^3 + y^3 = 3axy$$

Differentiating above equation with respect to x,

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3axy)$$

$$\Rightarrow \quad 3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx} \qquad \text{... (1)}$$

$$\Rightarrow \quad 3y^2.\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax}$$

$$\left.\frac{dy}{dx}\right|_{p\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{3a\left(\frac{3a}{2}\right) - 3\left(\frac{3a}{2}\right)^2}{3\left(\frac{3a}{2}\right)^2 - 3a\left(\frac{3a}{2}\right)}$$

$$= \frac{\frac{9a^2}{2} - \frac{27a^2}{4}}{\frac{27a^2}{4} - \frac{9a^2}{2}}$$

$$= \frac{18a^2 - 27a^2}{27a^2 - 18a^2}$$

$$= \frac{-9a^2}{9a^2}$$

$$= -1$$

Equation (1) can be written as,

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow \quad x^2 + y^2\frac{dy}{dx} = a\left(x\frac{dy}{dx} + y\right) \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x,

$$2x + y^2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)2y\left(\frac{dy}{dx}\right) = a\left[x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1)\frac{dy}{dx}\right]$$

$$\Rightarrow \quad y^2\left(\frac{d^2y}{dx^2}\right) - ax\frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - 2x - 2y\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right) = 2a\frac{dy}{dx} - 2x - \frac{2y\left(\frac{dy}{dx}\right)^2}{(y^2 - ax)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{-2a-2\left(\frac{3a}{2}\right)-2\left(\frac{3a}{2}\right)(-1)^2}{\frac{9a^2}{4}-a\left(\frac{3a}{2}\right)}$$

$$= \frac{-2a-3a-3a}{\frac{9a^2-6a^2}{4}}$$

$$= \frac{-32a}{3a^2}$$

$$= \frac{-32}{3a}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{-32}{3a}$$

$$\text{Curvature at } \left(\frac{3a}{2},\frac{3a}{2}\right) = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{\frac{-32}{3a}}{[1+(-1)^2]^{\frac{3}{2}}}$$

$$= \frac{\frac{-32}{3a}}{(2)^{\frac{3}{2}}}$$

$$= \frac{-32}{3a(2\sqrt{2})}$$

$$= \frac{-8\sqrt{2}}{3a}$$

$$\therefore \quad \text{Curvature at } \left(\frac{3a}{2},\frac{3a}{2}\right) = \frac{-8\sqrt{2}}{3a}.$$

Q17. Find the radius of curvature at any point on the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Answer :

Given that,

The parametric equations of equation (1) are,

$x = a\cos^3\theta$ and $y = a\sin^3\theta$

Differentiating the equations with respect to 'θ' on both sides,

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta),\ \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\Rightarrow \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

Consider,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{-\sin\theta}{\cos\theta}$$

$$\frac{dy}{dx} = -\tan\theta \quad \text{... (2)}$$

Differentiating equation (2) with respect to 'x',

$$\frac{d^2y}{dx^2} = -\sec^2\theta \times \frac{d\theta}{dx} = -\sec^2\theta \times \frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2\theta}{3a\cos^2\theta\sin\theta} \quad \text{... (3)}$$

The radius of curvature for a curve at any point is given as,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}$$

$$= \frac{(1+\tan^2\theta)^{3/2}}{\frac{\sec^2\theta}{3a\cos^2\theta\sin\theta}}$$

[∵ From equation (2) and (3)]

$$= \frac{3a\cos^2\theta\sin\theta\times(\sec^2\theta)^{3/2}}{\sec^2\theta}$$

$$= \frac{3a\cos^2\theta\sin\theta\sec^3\theta}{\sec^2\theta}$$

$$= 3a\cos^2\theta\sin\theta\sec\theta$$

$$= 3a\cos^2\theta\sin\theta \times \frac{1}{\cos\theta}$$

$$\therefore \quad \rho = 3a\cos\theta\sin\theta$$

Q18. Find the radius of curvature of the curve $x = a\cos^3t$, $y = b\sin^3t$ at $t = \pi/4$.

Answer :

Given curve is,

$$x = a\cos^3t \quad \text{... (1)}$$

$$y = b\sin^3t \quad \text{... (2)}$$

Differentiating equation (1) with respect to t.

$$\frac{dx}{dt} = a.3\cos^2t(-\sin t)$$

$$\Rightarrow \frac{dx}{dt} = -3a\sin t\cos^2t$$

Differentiating equation (2) with respect to 't'

$$\frac{dy}{dt} = b.3\sin^2t\cos t$$

$$\Rightarrow \frac{dy}{dt} = 3b\cos t\sin^2t$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b\cos t\sin^2t}{-3a\sin t\cos^2t} = \frac{-b\sin t}{a\cos t}$$

$$= \frac{-b}{a}\tan t$$

$$\therefore \quad \frac{dy}{dx} = \frac{-b}{a}\tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{b}{a}\tan t\right) = -\frac{b}{a}\cdot\frac{d}{dx}(\tan t)$$

$$= -\frac{b}{a}\cdot\frac{d}{dt}(\tan t)\cdot\frac{dt}{dx}$$

$$= -\frac{b}{a}\cdot\sec^2 t\cdot\frac{1}{dx/dt}$$

$$= -\frac{b}{a}\cdot\sec^2 t\cdot\frac{1}{-3a\cos^2 t\sin t}$$

$$= \frac{b\sec^4 t}{3a^2\sin t}$$

The expression for radius of curvature is given as,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Substituting the corresponding values in above equation,

$$\rho = \frac{\left[1+\left(-\frac{b}{a}\tan t\right)^2\right]^{\frac{3}{2}}}{\frac{b\sec^4 t}{3a^2\sin t}}$$

$$\rho = \frac{\left[1+\frac{b^2\tan^2 t}{a^2}\right]^{\frac{3}{2}}}{\frac{b\sec^4 4}{3a^2\sin t}} = \frac{\frac{(a^2+b^2\tan^2 t)^{\frac{3}{2}}}{a^3}}{\frac{b}{3a^2\sin t\cos^4 t}}$$

$$= \frac{(a^2+b^2\tan^2 t)^{\frac{3}{2}}.3a^2\sin t\cos 4t}{a^3 b}$$

$$\rho\Big|_{t=\frac{\pi}{4}} = \frac{\left[a^2+b^2\tan^2\frac{\pi}{4}\right]^{\frac{3}{2}}3a^2\sin\frac{\pi}{4}\cos^4\frac{\pi}{4}}{a^3 b}$$

$$= \frac{(a^2+b^2)^{\frac{3}{2}}3a^2.\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^4}{a^3 b}$$

$$= \frac{3(a^2+b^2)^{\frac{3}{2}}}{4\sqrt{2}\,ab}$$

$$\therefore \quad \rho = \frac{3(a^2+b^2)^{\frac{3}{2}}}{4\sqrt{2}\,.ab}.$$

Q19. Find the radius of curvature for the curve y = x^2 – 6x + 10 at (3, 1).

Answer :

Given that,

Equation of curve is,

$$y = x^2 - 6x + 10 \qquad \text{... (1)}$$

The expression for radius of curvature is given as,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \qquad \text{... (2)}$$

Differentiating equation (1) with respect to 'x',

$$\frac{dy}{dx} = 2x - 6 \qquad \text{... (3)}$$

Differentiating equation (3) with respect to 'x',

$$\frac{d^2y}{dx^2} = 2 \qquad \text{... (4)}$$

Substituting equations (3) and (4) in equation (2),

$$\rho = \frac{[1+(2x-6)^2]^{3/2}}{2}$$

$$= \frac{[1+4x^2+36-24x]^{3/2}}{2}$$

$$\rho = \frac{[4x^2-24x+37]^{3/2}}{2}$$

$$\rho_{(3,\,1)} = \frac{[4(3)^2-24(3)+37]^{3/2}}{2}$$

$$= \frac{[36-72+37]^{3/2}}{2}$$

$$= \frac{(1)^{3/2}}{2}$$

$$\therefore \rho_{(3,1)} = \frac{1}{2}$$

Q20. Find the radius of curvature at any point of s = c log sec ψ.

Answer :

Given,

$$s = c\log\sec\psi$$

The radius of curvature at any point is,

$$\rho = \frac{ds}{d\psi}$$

$$\Rightarrow \quad \rho = \frac{d}{d\psi}(c\log\sec\psi)$$

$$= c\,\frac{d}{d\psi}(\log(\sec\psi))$$

$$= c\,\frac{1}{\sec\psi}(\sec\psi\tan\psi)$$

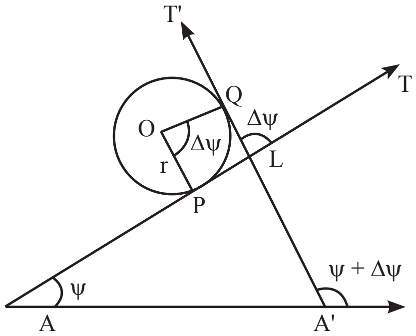
$$= c\tan\psi$$

$\therefore$ Radius of curvature is,

$$\rho = c\tan\psi.$$

Q21. Prove that the curvature of a circle is constant.

Answer :



Consider a circle with centre 'O' and radius 'r'.

Let P, Q be the points on the circle and let arc $PQ = \Delta s$.

Let 'L' be the point where the tangents PT, QT' at P and Q meet.

$$\angle POQ = \angle TLT' = \Delta\psi$$

From the sector formula, $l = r\theta$

$$\text{arc}(PQ) = (\angle POQ)(OP)$$

$$\Rightarrow \frac{\text{arc}(PQ)}{OP} = \angle POQ$$

$$\Rightarrow \frac{\Delta s}{r} = \Delta\psi \Rightarrow \frac{\Delta\psi}{\Delta s} = \frac{1}{r}$$

Let $Q \to P$ so that in the limit $\frac{d\psi}{ds} = \frac{1}{r}$.

Thus, the curvature at any point of a circle is the reciprocal of the radius of the circle and it is constant.

$\therefore$ The curvature of the circle is constant.

Q22. Define envelope.

Answer :

The curve which touches all the members of family of curves is defined as the envelope to that family of curves.

Q23. Obtain the equation of envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation ab = 4.

Answer : June/July-17, Q6

Given family of straight line is,

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{... (1)}$$

Where,

a, b are connected by the relation, $ab = 4$ ... (2)

Taking log on both sides of equation (2)

$$\log(ab) = \log 4$$

$$\Rightarrow \log a + \log b = 2\log 2 \quad \text{... (3)}$$

Let a and b are the functions of parameter 't',

Differentiating equations (1) and (3) with respect to 't',

$$\frac{-x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0$$

$$\Rightarrow \frac{x}{a^2}\frac{da}{dt} + \frac{y}{b^2}\frac{db}{dt} = 0 \quad \text{... (4)}$$

$$\frac{1}{a}\frac{da}{dt} + \frac{1}{b}\frac{db}{dt} = 0 \quad \text{... (5)}$$

Equating the terms of equations (4) and (5)

$$\Rightarrow \frac{\frac{x}{a^2}}{\frac{1}{a}} = \frac{\frac{y}{b^2}}{\frac{1}{b}}$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} = \frac{\frac{x}{a} + \frac{y}{b}}{1+1} = \frac{1}{1+1}$$

$$\Rightarrow a = \frac{x(2)}{1}, b = \frac{y(2)}{1}$$

$$\Rightarrow a = 2x,\ b = 2y$$

Substituting the values of a and b in equation (2)

$$\Rightarrow (2x)(2y) = 4$$

$$\Rightarrow xy = 1$$

Q24. Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where a + b = c, c is a constant.

Answer :

The given family of straight lines is,

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{... (1)}$$

Where,

$$a + b = c \quad \text{... (2)}$$

Differentiating equation (1) with respect to 't',

$$\Rightarrow x\left(\frac{-1}{a^2}\right)\frac{da}{dt} + y\left(\frac{-1}{b^2}\right)\frac{db}{dt} = 0$$

$$\Rightarrow \frac{-x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0$$

$$\Rightarrow \frac{x}{a^2}\frac{da}{dt} = \frac{-y}{b^2}\frac{db}{dt} \quad \text{... (3)}$$

Differentiating equation (2) with respect to 't',

$$\frac{da}{dt} + \frac{db}{dt} = 0$$

$$\frac{da}{dt} = \frac{-db}{dt} \quad \text{... (4)}$$

Dividing equation (3) by equation (4) and equating it to 1/c,

$$\frac{\frac{x}{a^2}\frac{da}{dt}}{\frac{da}{dt}} = \frac{\frac{-y}{b^2}\frac{db}{dt}}{\frac{-db}{dt}} = \frac{1}{c}$$

$$\Rightarrow \frac{x}{a^2} = \frac{y}{b^2} = \frac{1}{c}$$

$\Rightarrow \quad \dfrac{x}{a^2} = \dfrac{1}{c}, \dfrac{y}{b^2} = \dfrac{1}{c}$

$\Rightarrow \quad a^2 = cx, b^2 = cy$

$\Rightarrow \quad a = (cx)^{1/2}, b = (cy)^{1/2}$

$\Rightarrow \quad a = c^{1/2} x^{1/2}, b = c^{1/2} y^{1/2}$

By substituting the values of a and b in equation (2),

$c^{1/2} x^{1/2} + c^{1/2} y^{1/2} = c$ ... (5)

Dividing both sides of equation (5) by 'c',

$$\frac{c^{1/2}x^{1/2} + c^{1/2}y^{1/2}}{c} = \frac{c}{c}$$

$c^{-1/2} x^{1/2} + c^{-1/2} y^{1/2} = 1$

$c^{-1/2} (x^{1/2} + y^{1/2}) = 1$

$x^{1/2} + y^{1/2} = c^{1/2}$

$\therefore \quad \sqrt{x} + \sqrt{y} = \sqrt{c}$, is the required envelope.

Q25 . Find the envelope of the curve my + m² x – 10 = 0 where m is a parameter.

Answer :

Given curve is,

$my + m^2x - 10 = 0$

$\Rightarrow \quad m^2x + my - 10 = 0$

The above equation is quadratic in parameter 'm' therefore the envelope is given by,

Discriminant = 0

$\Rightarrow \quad (y)^2 - 4(x)(-10) = 0$

$\Rightarrow \quad y^2 + 40x = 0$

$\therefore$ Envelope of given curve is, $y^2 + 40x = 0$.

Q26. Find the envelope of the family of the lines y = mx + amᵖ, parameter being m.

Answer :

Given,

$y = mx + am^p$ ... (1)

Differentiating equation (1) partially with respect to 'm',

$0 = 1.x + a\,pm^{p-1}$

$\Rightarrow \quad apm^{p-1} = -x$

$\Rightarrow \quad m^{p-1} = \dfrac{-x}{ap}$ ... (2)

From equation (1),

$y = m(x + am^{p-1})$ ... (3)

Substituting equation (2) in equation (3),

$\Rightarrow y = m\left[x + a\left(\dfrac{-x}{ap}\right)\right]$

$\Rightarrow y = m\left[x - \dfrac{x}{p}\right]$

$\Rightarrow y = mx\dfrac{(p-1)}{p}$

$\Rightarrow yp = mx(p-1)$

$\Rightarrow (yp)^{p-1} = [mx(p-1)]^{p-1}$

$\Rightarrow y^{p-1}p^{p-1} = m^{p-1}x^{p-1}(p-1)^{p-1}$

$\Rightarrow y^{p-1}p^{p-1} = \left(\dfrac{-x}{ap}\right)x^{p-1}(p-1)^{p-1}$

$\Rightarrow ay^{p-1}p^{p-1+1} = -x^{p-1+1}(p-1)^{p-1}$

$\Rightarrow ay^{p-1}p^{p} = -(p-1)^{p-1}x^{p}$

$\Rightarrow ay^{p-1}p^{p} + (p-1)^{p-1}x^{p} = 0$

$\therefore$ The envelope is,

$ay^{p-1}p^{p} + (p-1)^{p-1}x^{p} = 0$

Q27. Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = a$ where α is a parameter.

Answer :

Given family of straight lines is,

$x \cos + y \sin = a$ ... (1)

Where,

α – Parameter.

Differentiating equation (1), with respect to 'α',

$-x \sin + y \cos = 0$ ... (2)

Squaring and adding equations (1) and (2),

$(x \cos + y \sin)^2 + (-x \sin + y \cos)^2 = a^2$

$\Rightarrow \quad x^2 \cos^2 + y^2 \sin^2 + 2\,xy \sin \cos + x^2 \sin^2 + y^2 \cos^2 - 2\,xy \sin \cos = a^2$

$\Rightarrow \quad x^2 [\sin^2 + \cos^2] + y^2 [\sin^2 + \cos^2] = a^2$

$\Rightarrow \quad x^2 + y^2 = a^2$

The required envelope of the given family of straight lines is, $x^2 + y^2 = a^2$

Q28 Find the envelope x² sin α + y² cos α = a², α is a parameter.

Answer :

Given,

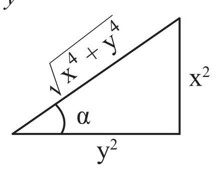
$x^2 \sin\alpha + y^2\cos\alpha = a^2$ ... (1)

Differentiating equation (1) with respect to 'α',

$x^2 \cos\alpha + y^2(-\sin\alpha) = 0$

$\Rightarrow \quad x^2 \cos\alpha = y^2\sin\alpha$

$\Rightarrow \quad \tan\alpha = \dfrac{x^2}{y^2}$



Figure

From the figure,

$$\sin\alpha = \frac{x^2}{\sqrt{x^4 + y^4}} \quad \text{... (2)}$$

$$\cos\alpha = \frac{y^2}{\sqrt{x^4 + y^4}} \quad \text{... (3)}$$

Substituting equations (2) and (3) in equation (1),

$$x^2\left(\frac{x^2}{\sqrt{x^4 + y^4}}\right) + y^2\left(\frac{y^2}{\sqrt{x^4 + y^4}}\right) = a^2$$

$$\Rightarrow \quad \frac{x^4}{\sqrt{x^4 + y^4}} + \frac{y^4}{\sqrt{x^4 + y^4}} = a^2$$

$$\Rightarrow \quad \left(x^4 + y^4\right)^{\frac{1}{2}} = a^2$$

$$\Rightarrow \quad x^4 + y^4 = a^4$$

$$\therefore \quad x^4 + y^4 = a^4.$$

Q29. Find the envelope of the family of circles $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2$ where α is the parameter.

Answer :

Given family of circles is,

$$x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2 \quad \text{... (1)}$$

$$\Rightarrow \quad x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha - c^2 = 0 \quad \text{... (2)}$$

$$\Rightarrow \quad 2ax\cos\alpha + 2ay\sin\alpha = x^2 + y^2 - c^2 \quad \text{... (3)}$$

Differentiating equation (2) with respect to α,

$$-2ax(-\sin\alpha) - 2ay\cos\alpha = 0$$

$$\Rightarrow \quad 2ax\sin\alpha - 2ay\cos\alpha = 0 \quad \text{... (4)}$$

Squaring and adding equations (3) and (4),

$$\Rightarrow \quad 4a^2x^2\cos^2\alpha + 4a^2y^2\sin^2\alpha + 8a^2xy\sin\alpha\cos\alpha + 4a^2x^2\sin^2\alpha + 4a^2y^2\cos^2\alpha - 8a^2xy\sin\alpha\cos\alpha = (x^2 + y^2 - c^2)^2$$

$$\Rightarrow \quad 4a^2\left[x^2(\sin^2\alpha + \cos^2\alpha) + y^2(\sin^2\alpha + \cos^2\alpha)\right] = (x^2 + y^2 - c^2)^2$$

$$\Rightarrow \quad 4a^2(x^2 + y^2) = (x^2 + y^2 - c^2)^2$$

$\therefore \quad 4a^2(x^2 + y^2) = (x^2 + y^2 - c^2)^2$ is the required envelope.

PART-B
ESSAY QUESTIONS WITH SOLUTIONS

2.1 ROLLE'S THEOREM, LAGRANGE'S, CAUCHY'S MEAN VALUE THEOREMS

Q30. State and prove Rolle's theorem. Explain the geometrical representation of the Rolle's theorem.

Answer :

Rolle's Theorem

For answer refer Unit-2, Q.No. 1.

Geometrical Representation of Rolle's Theorem

Rolle's theorem geometrically interprets that the curve $y = f(x)$ is such that,

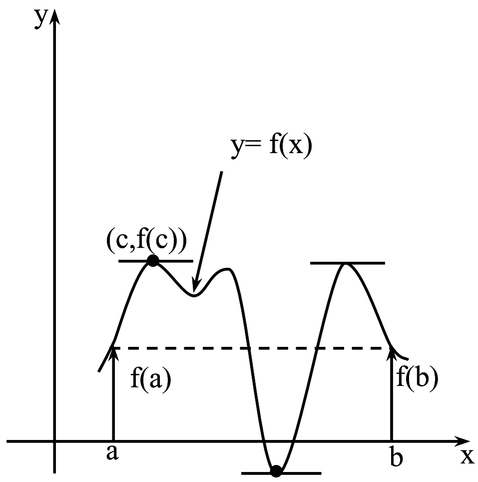
(i) It is continuous in the close interval $[a, b]$

(ii) It has a unique tangent to the curve at every point $(c, f(c))$ where $a < c < b$ and

(iii) The ordinates corresponding to $x = a$ and $x = b$ and equal i.e., $f(a) = f(b)$

By taking geometric interpretation into consideration, Rolle's theorem states that there is atleast one point on 'c'in (a,b) such that the tangent is parallel to x–axis.

Figure illustrates the diagrammatic representation of the geometric interpretation of Rolle's theorem.



Figure

Q31. If Rolle's mean value theorem holds for the function f(x) = $x^3 + ax^2 + bx$, $1 \le x \le 2$ at the point x = $\frac{4}{3}$ then find the values of a and b.

Answer :

Given,

Rolle's theorem holds for the function, $f(x) = x^3 + ax^2 + bx$, $1 \le x \le 2$.

Here, $x \in [1, 2]$

Rolle's theorem is satisfied only when $f(a) = f(b)$.

$\therefore \quad f(a) = f(1) = (1)^3 + a(1)^2 + b(1)$

$\Rightarrow \quad f(1) = a + b + 1$

$\therefore \quad f(b) = f(2) = (2)^3 + a(2)^2 + b(2)$

$= 8 + 4a + 2b$

$f(1) = f(2)$

$\Rightarrow \quad a + b + 1 = 8 + 4a + 2b$

$\Rightarrow \quad 2b - b = -8 + 1 - 4a + a$

$b = -3a - 7$

And, $x = \frac{4}{3}$

$\Rightarrow \quad f'(x) = 3x^2 + 2ax + b$

$\Rightarrow \quad f'\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right)^2 + 2a\left(\frac{4}{3}\right) + b = 0$

$\Rightarrow \quad \frac{16}{3} + \frac{8a}{3} + b = 0$

$\Rightarrow \quad 16 + 8a + 3b = 0$ ... (2)

Substituting the value of 'b' in equation (2),

$16 + 8a + 3(-7 - 3a) = 0$

$\Rightarrow \quad 16 + 8a - 21 - 9a = 0$

$\Rightarrow \quad -5 = a$

$\therefore \quad a = -5$

Substituting a value in equation (1),

$b = -3(-5) - 7$

$\Rightarrow \quad b = 8$

$\therefore \quad b = 8.$

Q32. Verify Rolle's theorem f(x) = x(x + 3) $e^{\frac{-x}{2}}$ in [– 3, 0].

Answer :

Given function is,

$$f(x) = x(x+3)e^{\frac{-x}{2}} \text{ in } [-3, 0]$$

(i) To Check Continuity of f(x)

$f(x)$ is continuous in the interval [– 3, 0]

(ii) To Check Differentiability of f(x)

$f(x)$ is derivable in the interval (– 3, 0)

$$f'(x) = \frac{d}{dx}\left(e^{\frac{-x}{2}}(x^2+3)\right)$$

$$= e^{\frac{-x}{2}}(2x+3) + (x^2+3)e^{\frac{-x}{2}}\left(\frac{-1}{2}\right)$$

$$\therefore \quad f'(x) = \frac{(-x^2+x+6)}{2}e^{\frac{-x}{2}}$$

(iii) To Check f(a) = f(b)

$$f(-3) = -3(-3+3)e^{-\left(\frac{-3}{2}\right)}$$

$$= 0$$

$$f(0) = 0(0+3)e^{\frac{-0}{2}} = 0$$

$f(0) = f(-3) = 0$

$\therefore \quad f(0) = f(-3)$

Hence, $f(x)$ satisfies all the three conditions of Rolle's theorem.

There exists a point $c \in [-3, 0]$ such that $f'(c) = 0$

$f'(c) = 0$

$\Rightarrow \quad \left(\frac{-c^2+c+6}{2}\right)e^{\frac{-c}{2}} = 0$

$\Rightarrow \quad -c^2 + c + 6 = 0$

$\Rightarrow \quad c^2 - c - 6 = 0$

$\Rightarrow \quad c(c-3) + 2(c-3) = 0$

$\Rightarrow \quad (c+2)(c-3) = 0$

$\therefore \quad c = -2, 3$

$c = 3 \notin (-3, 0)$

$\therefore \quad c = -2 \in (-3, 0)$

Hence, Rolle's theorem is verified.

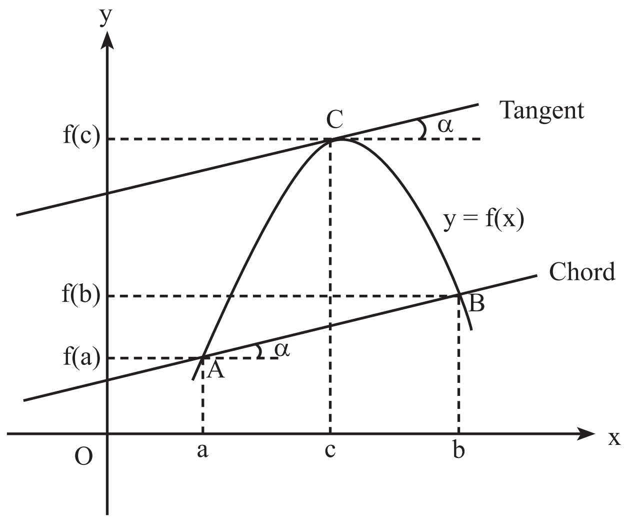
Q33. State Lagrange mean value theorem and give its geometrical interpretation.

Answer :

(i) Lagrange's Mean Value Theorem

For answer refer Unit-2, Q.No. 14.

Geometrical Representation of Lagrange's Mean Value Theorem



Figure

The chord is passing through the point of graph corresponding to the end segments a and b.

The slope (k) of chord $\overline{AB}$ = $\tan\alpha = \frac{f(b) - f(a)}{b - a}$

$f'(c)$ = Slope of tangent line at $C(c, f(c))$

$\Rightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a}$

Then, there exist a point $x = c$ inside the interval $[a, b]$ where the tangent to the graph is parellel to the chord

$\therefore \quad \tan\alpha = \frac{f(b) - f(a)}{b - a}, c \in [a, b].$

Q34. Show that $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$, 0 < u < v and deduce that $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$

Answer :

Consider the function as,

$f(x) = \tan^{-1} x$ defined in $[u, v]$ for $0 < a < b < 1$.

Since, $f(x)$ is continuous in closed interval $[u, v]$ and derivable in open interval (u, v), we can apply Lagrange's mean value theorem.

Hence, there exists a point $c \in [u, v]$ or $c \in (u, v)$

$$\therefore \quad f'(x) = \frac{f(v)-f(u)}{v-u} \quad \text{... (1)}$$

$$\text{Here, } f'(x) = \frac{1}{1+x^2} \quad \left[\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad f'(c) = \frac{1}{1+c^2} \quad \text{... (2)}$$

Equating the equations (1) and (2), we get,

$$\frac{1}{1+c^2} = \frac{f(v)-f(u)}{v-u} \quad \text{... (3)}$$

$c \in (u, v), 0 < u < v < 1.$

$\Rightarrow \quad u < c < v \Rightarrow \quad u^2 < c^2 < v^2$

$\Rightarrow \quad 1 + u^2 < 1 + c^2 < 1 + v^2$

$$\Rightarrow \quad \frac{1}{1+v^2} < \frac{1}{1+c^2} < \frac{1}{1+u^2}$$

$$\Rightarrow \quad \frac{1}{1+v^2} < \frac{f(v)-f(u)}{v-u} < \frac{1}{1+u^2}$$

[∵ From equation (3)]

$$\Rightarrow \quad \frac{v-u}{1+v^2} < f(v) - f(u) < \frac{v-u}{1+u^2}$$

$$\therefore \quad \frac{v-u}{1+v^2} < \tan^{-1}(v) - \tan^{-1}(u) < \frac{v-u}{1+u^2} \quad \text{... (4)}$$

$$\left[\because \begin{array}{l} f(x) = \tan^{-1}x \\ f(v) = \tan^{-1}v \\ f(u) = \tan^{-1}u \end{array}\right]$$

Let $u = 1, v = \frac{4}{3}$

Consider,

$$\frac{v-u}{1+v^2} = \frac{\frac{4}{3}-1}{1+\frac{16}{9}} = \frac{3}{25}$$

Consider,

$$\frac{v-u}{1+v^2} = \frac{\frac{4}{3}-1}{1+1} = \frac{1}{6}$$

$$f(v) = \tan^{-1}v = \tan^{-1}\left(\frac{4}{3}\right)$$

$$f(u) = \tan^{-1}u = \tan^{-1}(1) = \frac{\pi}{4}$$

Substituting the corresponding values in equation (4),

$$\frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$$

$$\therefore \quad \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}.$$

Q35. Verify Lagrange's Mean Value Theorem for $f(x) = x(x-1)(x-2);\ x \in \left[0, \frac{1}{2}\right]$.

Answer : Dec.-17, Q13(a)

Given function is,

$$f(x) = x(x-1)(x-2)$$
$$= x(x^2 - 3x + 2)$$
$$= x^3 - 3x^2 + 2x$$

(i) $f(x)$ is continuous in $\left(0, \frac{1}{2}\right)$

Since $f(x)$ is a polynomial in x

(ii) $f(x)$ is derivable in $\left(0, \frac{1}{2}\right)$

Since $f'(x) = 3x^2 - 6x + 2$ is defined in $\left(0, \frac{1}{2}\right)$

(iii) From Lagrange's theorem, there exists

$c \in \left(0, \frac{1}{2}\right)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{... (1)}$$

Here,

$$f(b) = f\left(\frac{1}{2}\right)$$
$$= \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)$$
$$\therefore \quad f\left(\frac{1}{2}\right) = \frac{3}{8}$$

$$f(a) = f(0)$$
$$= (0)^3 - 3(0)^2 + 2(0)$$
$$= 0$$
$$f(0) = 0$$
$$f'(c) = 3c^2 - 6c + 2$$

Substituting the corresponding values in equation (1),

$$3c^2 - 6c + 2 = \frac{\frac{3}{8}-0}{\frac{1}{2}-0}$$

$$\Rightarrow \quad 3c^2 - 6c + 2 - \frac{3}{4} = 0$$

$$\Rightarrow \quad 3c^2 - 6c + \frac{5}{4} = 0$$

$$\Rightarrow \quad c = \frac{-(-6) \pm \sqrt{(36) - 4(3)\frac{5}{4}}}{6}$$

$$\Rightarrow \quad c = \frac{6 \pm \sqrt{21}}{6}$$

Since $\frac{6 \pm \sqrt{21}}{6} = 1.76, \frac{6-\sqrt{21}}{6} = 0.24 \in \left(0, \frac{1}{2}\right)$

$\therefore \quad c = \frac{6-\sqrt{21}}{6}$

Q36. State Cauchy's mean value theorem and verify if for the functions f(x) = e^{-x} and g(x) = e^{x} in [a, b].

Answer : **Dec.-16, Q13(a)**

Statement

For answer refer Unit-2, Q7, Topic: Statement.

(ii) Given functions are,

$f(x) = e^{-x},\ g(x) = e^{x}$ in $[a, b]$

$f'(x) = -e^{-x}\ ;\ g'(x) = e^{x}$

$f(x)$ and $g(x)$ are continuous and derivable on (a, b).

Hence, according to cauchy's mean value theorem, there must be some $c \in [a, b]$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \quad \frac{-e^{-c}}{e^{c}} = \frac{e^{-b}-e^{-a}}{e^{b}-e^{a}}$$

$$\Rightarrow \quad \frac{-1}{e^{c}.e^{c}} = \frac{\frac{1}{e^{b}}-\frac{1}{e^{a}}}{e^{b}-e^{a}}$$

$$\Rightarrow \quad \frac{-1}{e^{2c}} = \frac{e^{a}-e^{b}}{e^{a+b}(e^{b}-e^{a})}$$

$$\Rightarrow \quad \frac{-1}{e^{2c}} = \frac{(e^{b}-e^{a})}{e^{a+b}(e^{b}-e^{a})}$$

$$\Rightarrow \quad \frac{1}{e^{2c}} = -\frac{1}{e^{a+b}}$$

$$\Rightarrow \quad 2c = a + b$$

$$\Rightarrow \quad c = \frac{a+b}{2} \in [a, b].$$

Q37. Find 'c' of the Cauchy's mean value theorem for the functions f(x) = $\frac{1}{x^2}$ and g(x) = $\frac{1}{x}$ in [a, b].

Answer :

The given functions are,

$f(x) = \frac{1}{x^2},\ g(x) = \frac{1}{x}$ in $[a, b]$

$\Rightarrow \quad f'(x) = \frac{-2}{x^3}$

$f(x)$ is continuous on $[a, b]$ and derivable on (a, b)

Also,

$g'(x) = \frac{-1}{x^2} \neq 0$

$g(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Hence, according to Cauchy's mean value theorem, there must be some $c \in [a, b]$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \quad \frac{-2/c^3}{-1/c^2} = \frac{1/b^2 - 1/a^2}{\frac{1}{b}-\frac{1}{a}}$$

$$\Rightarrow \quad \frac{2}{c} = \frac{a^2-b^2}{a^2b^2}\cdot\frac{ab}{a-b} = \frac{a+b}{ab}$$

$$\therefore \quad c = \frac{2ab}{a+b} \in [a, b]$$

Hence, Cauchy's mean value theorem is verified.

Q38. If f(x) = $\sqrt{x}$ and g(x) = $\frac{1}{\sqrt{x}}$ prove that 'c' of the Cauchy's generalized mean value theorem is the geometric mean of 'a' and 'b' for any a > 0, b > 0.

Answer :

The conditions of Cauchy's generalized mean value theorem to be satisfied by the given functions $f(x)$ and $g(x)$ as mentioned below,

Condition 1

$f(x)$ and $g(x)$ are continuous in $[a, b]$ and

$$f(x) = \sqrt{x} \text{ and } g(x) = \frac{1}{\sqrt{x}} \quad \ldots (1)$$

Condition 2

$$\left.\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \text{ and} \\ g'(x) &= \frac{-1}{2x\sqrt{x}} \end{aligned}\right\} \quad \ldots (2)$$

Hence, $f'(x)$ and $g'(x)$ exists in (a, b)

$\therefore \quad f(x)$ and $g(x)$ are differentiable in (a, b)

Condition 3

Also, $g'(x) \neq 0\ \forall\ x \in (a, b)$

Hence, all the three conditions of Cauchy's mean value theorem are satisfied.

If these conditions are satisfied then there exists a point $c \in (a, b)$ such that,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \ldots (3)$$

$\therefore \quad$ By substituting equations (1) and (2) in equation (3),

$$\Rightarrow \quad \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}}$$

$$\Rightarrow \quad \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}}$$

$$\Rightarrow \quad \frac{-(\sqrt{a}-\sqrt{b})(\sqrt{ab})}{(\sqrt{a}-\sqrt{b})} = -c$$

$$\Rightarrow \quad \sqrt{ab} = c$$

$\therefore$ c of the Cauchy's generalized mean value theorem is the geometric mean of 'a' and 'b' for $a > 0$, $b > 0$.

2.2 TAYLOR'S SERIES

Q39. State and prove Taylor's theorem.

Answer :

For answer refer Unit-2, Q9.

Hypothesis

$f(x), f'(x), f''(x)$ are continuous in $[a, a+h]$

Proof

Consider the function,

$$\phi(x) = f(x) + \frac{(a+h-x)}{1!} f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots + \frac{(a+h-x)^n}{n!} K \qquad \ldots (1)$$

Where, K is constant (unknown).

Substituting $x = a + h$ in equation (1),

$$\phi(a+h) = f(a+h) + (a+h-(a+h))f'(x) + \frac{(a+h-(a+h))^2}{2!} f''(x) + \ldots \frac{(a+h-(a+h))^n}{n!} K$$

$$= f(a+h) + 0 + 0 \ldots + 0 \qquad \ldots (2)$$

Substituting $x = a$ in equation (1),

$$\phi(a) = f(a) + \frac{(a+h-a)}{1!} f'(a) + \frac{(a+h-a)^2}{2!} f''(a) + \ldots + \frac{(a+h-a)^n}{n!} K$$

$$= f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots \frac{h^n}{n!} K$$

K is defined by,

$$\phi(a) = \phi(a+h)$$

$\therefore$ Equating equations (1) and (2),

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots \frac{h^n}{n!} K \qquad \ldots (3)$$

The equation (3) satisfies Rolle's theorem i.e.,

1. New function ϕ is continuous in $[a, a+h]$
2. Derivable in $(a, a+h)$
3. $\phi(a) = \phi(a+h)$

Then there exists one number c between a and $a + h$.

$\therefore \phi'(c) = \phi'(a + \theta h) = 0$

$\therefore \quad K = f^n(a + \theta h) \qquad (0 < \theta < 1)$

Substituting 'K' value in equation (3), we get,

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots \frac{h^n}{n!} f^n(a + \theta h)$$

Q40. Expand $2x^3 + 7x^2 + x - 6$ in powers of $(x - 2)$ by using Taylor's theorem.

Answer :

Given that,

$$f(x) = 2x^3 + 7x^2 + x - 6 \quad \text{... (1)}$$

$$f(2) = 16 + 28 + 2 - 6 = 40$$

Differentiating equation (1) with respect to 'x', to get the required terms as f', f'' and f'''.

$$f'(x) = 6x^2 + 14x + 1$$

$$\Rightarrow \quad f'(2) = 24 + 28 + 1 = 53$$

$$f''(x) = 12x + 14$$

$$\Rightarrow \quad f''(2) = 24 + 14 = 38$$

$$f'''(x) = 12,$$

$$\Rightarrow \quad f'''(2) = 12$$

By Taylor's series,

$$f(x) = f(a) + \frac{h^1}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + ...$$

Substituting

$$a = 2;\ h = x - 2$$

$$f(x) = f(2) + \frac{(x-2)^1}{1!}f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f''(2) \quad \text{... (2)}$$

Substituting $f(a), f'(a), f''(a), f'''(a)$ values in equation (2),

$$\Rightarrow \quad f(x) = 40 + \frac{(x-2)^1}{1}(53) + \frac{(x-2)^2}{2}(38) + \frac{(x-2)^3}{6}(12)$$

$$\therefore \quad f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

Q41. Write Taylor's series for $f(x) = (1 - x)^{5/2}$ with Lagrange's form of remainder upto 3 terms in the interval [0, 1].

Answer :

Given that,

$$f(x) = (1 - x)^{5/2} \text{ in } [0, 1] \quad \text{... (1)}$$

Here, $a = 0$ and $h = 1$

$$\Rightarrow \quad f(a) = (1 - a)^{5/2}$$

$$\Rightarrow \quad f(0) = (1 - 0)^{5/2}$$

$$\Rightarrow \quad f(0) = 1$$

Taylor's series is given as,

$$f(a + h) = f(a) + h f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + ... \quad \text{... (2)}$$

Differentiating equation (1) with respect to 'x',

$$f'(x) = \frac{5}{2}(1-x)^{3/2}(-1) \Rightarrow f'(0) = \frac{-5}{2}$$

$$\Rightarrow \quad f''(x) = \left(-\frac{5}{2}\right)\left(\frac{3}{2}\right)(1-x)^{1/2}(-1) \quad ; \quad f''(0) = \frac{15}{4}$$

$$\Rightarrow \quad f'''(x) = \left(\frac{15}{4}\right)\left(\frac{1}{2}\right)(1-x)^{-1/2}(-1) \quad ; \quad f'''(0) = \frac{-15}{8}$$

Substituting the above values in equation (2),

$$f(0+1) = 1+1\left(\frac{-5}{2}\right)+\frac{1}{2}\left(\frac{15}{4}\right)+\frac{1}{6}\left(-\frac{15}{8}\right) \quad [\because a=0, h=1]$$

$$\therefore \quad f(1) = 1-\frac{5}{2}+\frac{15}{8}-\frac{5}{16}+.....$$

Q42. Find the taylor series expansion of f(x) = x³ + 3x² + 2x + 1 about x = –1.

Answer :

Given that functions is,

$f(x) = x^3 + 3x^2 + 2x + 1$

$x = -1$

Taylor's series expansion of $f(x)$ about $x = a$ is given as,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)+....$$

Here,

$a = -1$

$$\Rightarrow \quad f(x) = f(-1) + (x+1)f'(-1) + \frac{(x+1)^2}{2!}f''(1)+.... \quad ... (1)$$

$f(x) = x^3 + 3x^2 + 2x + 1$

$f(-1) = (-1)^3 + 3(-1)^2 + 2(-1) + 1$

$= -1 + 3 - 2 + 1 = 1$

$f'(x) = 3x^2 + 6x + 2$

$f'(-1) = 3(-1)^2 + 6(-1) + 2$

$= 3 - 6 + 2 = -1$

$f''(x) = 6x + 6$

$f''(-1) = 6(-1) + 6$

$= 0$

$f'''(x) = 6$

$f'''(-1) = 6$

Substituting the corresponding values in equation (1),

$$f(x) = 1 + (x+1)(-1) + \frac{(x+1)^2}{2!}(0)+\frac{(x+1)^3}{3!}(6)+....$$

$\Rightarrow \quad f(x) = 1 - (x+1) + 0 + (x+1)^3$

$\therefore \quad f(x) = -x + (x+1).$

2.3 CURVATURE, RADIUS OF CURVATURE, CIRCLE OF CURVATURE

Q43. Find the radius of curvature for the curve r = a(1 – cosθ).

Answer :

Given equation is,

$$r = a(1-\cos\theta) \quad ... (1)$$

The radius of curvature is given as,

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1-rr_2}$$

Where,

$$r_1 = \frac{dr}{d\theta}, r_2 = \frac{d^2r}{d\theta^2}$$

$$r^2 = a^2(1-\cos\theta)^2 \qquad [\because \text{ From equation (1)}]$$

$$r_1 = \frac{d}{d\theta} a(1-\cos\theta) = a[-(-\sin\theta)] = a\sin\theta$$

$$r_1 = a\sin\theta, \ r_1^2 = a^2\sin^2\theta$$

$$r_2 = \frac{d}{d\theta}(a\sin\theta) = a\cos\theta$$

Substituting values of r^2, r_1^2, r_2, r in equation (1),

$$\rho = \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{\frac{3}{2}}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta)(a\cos\theta)}$$

$$= \frac{(a^2)^{\frac{3}{2}}[1+\cos^2\theta - 2\cos\theta + \sin^2\theta]^{\frac{3}{2}}}{a^2[1+\cos^2\theta - 2\cos\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta]}$$

$$= \frac{a^3[2-2\cos\theta]^{\frac{3}{2}}}{a^2[1-3\cos\theta + 2(\sin^2\theta + \cos^2\theta)]}$$

$$= \frac{a^3[2(1-\cos\theta)]^{\frac{3}{2}}}{a^2[3(1-\cos\theta)]^{\frac{3}{2}}}$$

$$= \frac{a[2(1-\cos\theta)]^{\frac{3}{2}}}{[3(1-\cos\theta)]^{\frac{3}{2}}}$$

$$= \frac{a[2(1-\cos\theta)]^{\frac{3}{2}}}{[3(1-\cos\theta)]^{\frac{3}{2}}}$$

$$= \frac{a[2]^{\frac{3}{2}}.(1-\cos\theta)^{\frac{3}{2}}}{(3)^{\frac{3}{2}}(1-\cos\theta)^{\frac{3}{2}}}$$

$$= \frac{a2\sqrt{2}}{3\sqrt{3}}$$

$$= \frac{2}{3}a\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$$

$$\therefore \ \rho = \frac{2\sqrt{2}}{3\sqrt{3}}a$$

Q44. If x = a(t + sint), y = a(1 – cost) then show that $\rho = 4a\cos\left(\frac{t}{2}\right)$.

Answer :

Given curves are,

$$x = a(t + \sin t) \qquad \text{... (1)}$$

$$y = a(1 - \cos t) \qquad \text{... (2)}$$

Differentiating equations (1) and (2) with respect to 't',

$$\frac{dx}{dt} = a(1+\cos t)$$

$$\frac{dy}{dt} = a(\sin t)$$

Consider,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a(\sin t)}{a(1+\cos t)}$$

$$= \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\cos^2\left(\frac{t}{2}\right)}$$

$$\therefore \ \frac{dy}{dt} = \tan\left(\frac{t}{2}\right)$$

Differentiating the above equation with respect to x,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\left(\frac{t}{2}\right)\right)$$

$$\Rightarrow \ \frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right).\frac{1}{2}\frac{dt}{dx}$$

$$= \frac{\frac{1}{2}\sec^2\left(\frac{t}{2}\right)}{a(1+\cos t)}$$

$$\Rightarrow \ \frac{d^2y}{dx^2} = \frac{\frac{1}{2}\sec^2\left(\frac{t}{2}\right)}{a.\cos^2\left(\frac{t}{2}\right)}$$

$$\therefore \ \frac{d^2y}{dx^2} = \frac{1}{4a\cos^4\left(\frac{t}{2}\right)}$$

The radius of curvature is given as,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \qquad \text{... (3)}$$

Substituting the corresponding values in equation (3),

$$= \frac{\left[1+\tan^2\left(\frac{t}{2}\right)\right]^{\frac{3}{2}}}{\frac{1}{4a\cos^4\left(\frac{t}{2}\right)}}$$

$$= \frac{\left(\sec^2\left(\frac{t}{2}\right)\right)^{\frac{3}{2}}}{\frac{1}{4a}.\frac{1}{\cos^4\left(\frac{t}{2}\right)}} = \frac{4a.\sec^3\left(\frac{t}{2}\right)}{\frac{1}{\cos^4\left(\frac{t}{2}\right)}}$$

$$\rho = 4a.\cos\left(\frac{t}{2}\right)$$

$$\therefore \ \rho = 4a.\cos\left(\frac{t}{2}\right).$$

Q45. Show that radius of curvature at any point of the astroid x = a cos³θ, y = a sin³θ is equal to three times the length of the perpendicular from the origin to the tangent.

Answer :

Given parametric equations of the astroid are,

$x = a\cos^3\theta,\ y = a\sin^3\theta$

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{3a\sin^2\theta\frac{d}{d\theta}(\sin\theta)}{3a\cos^2\theta\frac{d}{d\theta}(\cos\theta)}$$

$$= \frac{3a\sin^2\theta.\cos\theta}{-3a\cos^2\theta.\sin\theta}$$

$\Rightarrow y_1 = -\tan\theta$

$$y_2 = \frac{d^2y}{dx^2}$$

$$= \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$$

$$= -\sec^2\theta\times\frac{1}{-3a\cos^2\theta.\sin\theta}$$

$$\Rightarrow y_2 = \frac{1}{3a\cos^4\theta.\sin\theta}$$

∴ The radius curvature 'ρ' at any point of given astroid is,

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

$$= \frac{(1+\tan^2\theta)^{\frac{3}{2}}}{\frac{1}{3a\cos^4\theta.\sin\theta}}$$

$$= (\sec^2\theta)^{\frac{3}{2}}\times[3a\cos^4\theta.\sin\theta]$$

$$= \sec^3\theta\times 3a\cos^4\theta\sin\theta$$

$$= \frac{1}{\cos^3\theta}\times 3a\cos^4\theta\sin\theta$$

$$= \frac{3a}{2}2\sin\theta\cos\theta$$

$$= \frac{3a}{2}\sin 2\theta$$

$$\therefore\ \rho = \frac{3a}{2}\sin 2\theta \qquad \text{... (1)}$$

The equation of tangent at any point on the curve is,

$$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$$

$$\Rightarrow\ y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$$

$$\Rightarrow\ y\cos\theta + x\sin\theta = a\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow\ y\cos\theta + x\sin\theta - a\sin\theta\cos\theta = 0 \qquad \text{... (2)}$$

The length of perpendicular from origin to the tangent is,

$$p = \frac{|0(\sin\theta) + 0(\cos\theta) - a\sin\theta.\cos\theta|}{\sqrt{\sin^2\theta+\cos^2\theta}}$$

$$p = a\sin\theta\cos\theta$$

$$= \frac{a}{2}\sin 2\theta$$

$$p = \frac{a}{2}\sin 2\theta \qquad \text{... (3)}$$

From equations (1) and (3),

$\rho = 3p$

∴ The radius of curvature of astroid is 3 times the length of perpendicular from origin to the tangent.

Q46. Find the radius of curvature of the curve x = a(θ – sin θ), y = a(1 – cos θ) at θ = π.

Answer :

The given curves are,

$$x = a(t - \sin t) \qquad \text{... (1)}$$

$$y = a(1 - \cos t) \qquad \text{... (2)}$$

The radius of curvature for a curve in parametric form; $x = x(t)$, $y = y(t)$ is given as,

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)} \qquad \text{... (3)}$$

$x = a(t - \sin t)$

Differentiating with respect to 't',

$$\frac{dx}{dt} = a(1-\cos t) \qquad \text{... (4)}$$

At $t = \pi$

$$\frac{dx}{dt} = a(1-\cos\pi)$$

$$= a - a\cos\pi$$

$$= a - a(-1)$$

$$= a + a$$

$$\frac{dx}{dt} = 2a \qquad \text{... (5)}$$

$$\frac{d^2x}{dt^2} = a\sin t$$

$$\therefore\ \frac{d^2x}{dt^2} = a\sin\pi = 0 \qquad \text{... (6)}$$

Differentiating equation (2) with respect to 't',

$$\frac{dy}{dt} = a\sin t$$

At $t = \pi$

$$\frac{dy}{dt} = a\sin\pi$$

$$\therefore\ \frac{dy}{dt} = 0 \qquad \text{... (7)}$$

$$\frac{d^2y}{dt^2} = a\cos t = a\cos\pi$$

$$\therefore \quad \frac{d^2y}{dt^2} = -a \quad \text{... (8)}$$

From equations (5) and (7),

$$\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 = (2a)^2 + 0 = 4a^2$$

$$\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2} = (4a^2)^{3/2}$$

$$= 8a^3 \quad \text{... (9)}$$

From equations (5), (6), (7) and (8),

$$\left(\frac{dx}{dt}\right)\times\left(\frac{d^2y}{dt^2}\right) - \left(\frac{d^2x}{dt^2}\right)\times\left(\frac{dy}{dt}\right) = (2a)(-a) - 0$$

$$= -2a^2 \quad \text{... (10)}$$

On substituting equations (9) and (10) in equation (3),

$$\rho = \frac{8a^3}{-2a^2} \qquad \text{[Neglect negative sign]}$$

$$\therefore \rho = 4a$$

Q47. Find the circle of curvature of the curve x = a (cos t + t sin t), y = a (sin t – t cos t) at $t = \frac{\pi}{4}$.

Answer :

Given that,

$$x = a(\cos t + t\sin t) \quad \text{... (1)}$$

$$y = a(\sin t - t\cos t) \quad \text{... (2)}$$

Differentiating equation (1) with respect to 't',

$$\frac{dx}{dt} = \frac{d}{dt}\Big(a(\cos t + t\sin t)\Big)$$

$$= \frac{d}{dt}(a\cos t) + \frac{d}{dt}(a\, t\sin t)$$

$$= a\frac{d}{dt}(\cos t) + a\frac{d}{dt}(t\sin t)$$

$$= a(-\sin t) + a(t\cos t) + a(1\sin t)$$

$$= -a\sin t + a\,t\cos t + a\sin t$$

$$\therefore \quad \frac{dx}{dt} = at\cos t$$

Differentiating equation (2) with respect to 't',

$$\frac{dy}{dt} = \frac{d}{dt}\Big[a(\sin t - t\cos t)\Big]$$

$$= \frac{d}{dt}(a\sin t) - \frac{d}{dt}(a\,t\cos t)$$

$$= a\frac{d}{dt}(\sin t) - a\frac{d}{dt}(t\cos t)$$

$$= a(\cos t) - a[t(-\sin t) + 1.(\cos t)]$$

$$= a\cos t + a\,t\sin t - a\cos t$$

$$\therefore \quad \frac{dy}{dt} = at\sin t$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t}$$

$$\therefore \quad \frac{dy}{dx} = \tan t = y_1 \quad \text{... (3)}$$

Differentiating equation (3) with respect to 'x',

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d\left[\frac{dy}{dx}\right]}{dt}\,\frac{dt}{dx}$$

$$= \frac{d(\tan t)}{dt}\cdot\frac{1}{at\cos t}$$

$$= \frac{\sec^2 t}{at\cos t}$$

$$= \frac{\sec^2 t\times\sec t}{at}$$

$$= \frac{\sec^3 t}{at}$$

$$\therefore \quad \frac{d^2y}{dx^2} = y_2 = \frac{\sec^3 t}{at} \quad \text{... (4)}$$

The equation of the circle of curvature is given as,

$$(x - X)^2 + (y - Y)^2 = \rho^2 \quad \text{... (5)}$$

$\therefore$ At (x, y),

$$\rho = \frac{(1 + y_1)^{\frac{3}{2}}}{y_2}$$

$$= \frac{\left[1 + (\tan t)^2\right]^{\frac{3}{2}}}{\left[\frac{\sec^3 t}{at}\right]} \quad [\because \text{From equations (3) and (4)}]$$

$$= \frac{(1 + \tan^2 t)^{\frac{3}{2}}}{(\sec^3 t)}\times at = \frac{\sec^3 t}{\sec^3 t}\times at$$

$$\therefore \quad \rho = at \quad \text{... (6)}$$

If (X, Y) are the coordinates of the centre then,

$$X = x - \frac{y_1}{y_2}(1 + y_1^2) \text{ and}$$

$$Y = y + \frac{1}{y_2}(1 + y_1^2)$$

$$\therefore \quad X = x - \frac{\tan t}{(\sec^3 t/at)}(1 + \tan^2 t)$$

$$= x - \frac{\tan t(at)}{\sec^3 t}(\sec^2 t)$$

$$= x - at\frac{(\tan t)}{\sec t}$$

$$= x - at\frac{(\sin t/\cos t)}{(1/\cos t)}$$

$$= x - at\left[\frac{\sin t}{\cos t}\right]\cos t$$

$$= x - at(\sin t)$$

$\therefore \quad X = a(\cos t + t\sin t) - at\sin t$

[$\because$ From equation (1)]

$= a\cos t + at\sin t - at\sin t$

$X = a\cos t \quad \text{... (7)}$

And,

$$Y = y + \frac{1}{(\sec^3 t/at)}(1+\tan^2 t)$$

$$= y + \frac{at}{\sec^3 t}(\sec^2 t)$$

$$= y + \frac{at}{\sec t}$$

$= y + a\,t\cos t$

$\therefore \quad Y = a(\sin t - t\cos t) + a\,t\cos t$

[$\because$ From equation (2)]

$= a\sin t - a\,t\cos t + a\,t\cos t$

$\therefore \quad Y = a\sin t \quad \text{...(8)}$

Substituting equations (6), (7) and (8) in equation (5),

$$[x-(a\cos t)]^2 + [y-(a\sin t)]^2 = (at)^2$$

$\Rightarrow \quad x^2 + a^2\cos^2 t - 2ax\cos t + y^2 + a^2\sin^2 t - 2a\,y\sin t = a^2t^2$

$\Rightarrow \quad x^2 + y^2 + a^2(\cos^2 t + \sin^2 t) - 2a(x\cos t + y\sin t) = a^2t^2$

$\Rightarrow \quad x^2 + y^2 + a^2(1) - 2a(x\cos t + y\sin t) = a^2t^2$

$\Rightarrow \quad x^2 + y^2 - 2a(x\cos t + y\sin t) = a^2t^2 - a^2$

$\Rightarrow \quad x^2 + y^2 - 2a(x\cos t + y\sin t) = a^2(t^2 - 1)$

The equation of the circle of curvature of equations (1) and (2) is given as,

$$x^2 + y^2 - 2a(x\cos t + y\sin t) = a^2(t^2 - 1)$$

At $t = \frac{\pi}{4}$,

$$x^2 + y^2 - 2a\left(x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4}\right) = a^2\left(\left(\frac{\pi}{4}\right)^2 - 1\right)$$

$\Rightarrow \quad x^2 + y^2 - \sqrt{2}\,a(x+y) = \frac{a^2(\pi^2 - 16)}{16}$

$\Rightarrow \quad 16x^2 + 16y^2 - 16\sqrt{2}\,a(x+y) = a^2(\pi^2 - 16)$

$\therefore$ The equation of the circle of curvature is,

$$16x^2 + 16y^2 - 16\sqrt{2}\,a(x+y) = a^2(\pi^2 - 16).$$

Q48. Find the circle of curvature of the curve $ay^2 = x^3$ at p(a, a).

Answer :

Given curve is,

$$ay^2 = x^3$$

Point, $p(x, y) = (a, a)$

$\Rightarrow \quad y^2 = \frac{x^3}{a} \quad \text{... (1)}$

Differentiating equation (1) with respect to x,

$$\frac{dy^2}{dx} = \frac{d}{dx}\left(\frac{x^3}{a}\right)$$

$\Rightarrow \quad 2y\frac{dy}{dx} = \frac{3x^2}{a}$

$\Rightarrow \quad 2y(y_1) = \frac{3x^2}{a} \qquad \left[\because \frac{dy}{dx} = y_1\right]$

$\Rightarrow \quad y_1 = \frac{3x^2}{2ay} \quad \text{... (2)}$

Differentiating equation (2) with respect to x,

$$\frac{d(y_1)}{dx} = \frac{d}{dx}\left[\frac{3}{2a}\left(\frac{x^2}{y}\right)\right]$$

$\Rightarrow \quad y_2 = \frac{3}{2a}\left[\frac{y\frac{d}{dx}(x^2) - x^2\left(\frac{d}{dx}\right)(y)}{y^2}\right]$

$$= \frac{3}{2a}\left[\frac{y.2x - x^2\frac{dy}{dx}}{y^2}\right]$$

$$y_2 = \frac{3}{2a}\left[\frac{2xy - x^2y_1}{y^2}\right]$$

From equation (2),

$$y_1|_{(a,a)} = \left(\frac{3x^2}{2ay}\right)_{(a,a)} = \frac{3a^2}{2aa} = \frac{3}{2}\cdot\frac{a^2}{a^2} = \frac{3}{2}$$

$\therefore \quad y_1|_{(a,a)} = \frac{3}{4a}$

$$y_2|_{(a,a)} = \left(\frac{3}{2a}\left(\frac{2xy - x^2y_1}{y^2}\right)\right)_{(a,a)}$$

$$= \frac{3}{2a}\left(\frac{2(a)(a) - (a)^2\frac{3}{2}}{a^2}\right)$$

$$= \frac{3}{2a}\left(\frac{4a^2 - 3a^2}{2a^2}\right)$$

$$= \frac{3}{2a}\left[\frac{a^2}{2a^2}\right]$$

$$= \frac{3}{4a}$$

$\therefore \quad y_2|_{(a,a)} = \frac{3}{4a}$

Q49. Find the circle of the curvature of the curve $y = x^2 - 6x + 10$ at P(3, 1).

Answer :

Let $(\bar{x}, \bar{y})$ be the centre of curvature. The expression for circle of curvature is given as,

$$(x - \bar{x})^2 + (y - \bar{y})^2 = (\rho)^2 \quad \text{... (1)}$$

Where, $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$\bar{y} = y + \frac{1+y_1^2}{y_2}$

$\bar{x} = x - \frac{(2x-6)[1+(2x-6)^2]}{2}$

$\bar{x}_{(3,1)} = 3 - \frac{(2(3)-6)[1+2(3)-6)]^2}{2}$

$= 3 - \frac{0(1+0)^2}{2}$

$= 3 - 0$

$= 3$

$\therefore \quad \bar{x} = 3$

$\bar{y} = y + \frac{1+(2x-6)^2}{2}$

$\bar{y}\,|_{(3,1)} = \frac{1+[2(3)-6]^2}{2}$

$= 1 + \frac{1+(0)^2}{2}$

$= 1 + \frac{1}{2} = \frac{3}{2}$

$\therefore \quad \bar{y} = \frac{3}{2}$

Substituting the values of $\bar{x}$, $\bar{y}$ and ρ in equation (5),

$(x-3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \qquad \left[\because \rho = \frac{1}{2}\right]$

$\Rightarrow \quad x^2 - 6x + 9 + y^2 - 3y + \frac{9}{4} = \frac{1}{4}$

$\Rightarrow \quad x^2 + y^2 - 6x - 3y = \frac{1}{4} - \frac{9}{4} - 9$

$\Rightarrow \quad x^2 + y^2 - 6x - 3y = \frac{1-9-36}{4}$

$\Rightarrow \quad x^2 + y^2 - 6x - 3y = \frac{-44}{4}$

$\Rightarrow \quad x^2 + y^2 - 6x - 3y = -11$

$\Rightarrow \quad x^2 + y^2 - 6x - 3y + 11 = 0$

$\therefore$ Equation of circle of curvature is, $x^2 + y^2 - 6x - 3y + 11 = 0$.

2.4 ENVELOPE OF A FAMILY OF CURVES

Q50. Find the envelope of $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$, α is a parameter.

Answer :

Given that,

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \qquad \text{... (1)}$$

Differentiating equation (1), with respect to 'α',

$$\left(\frac{x}{a}\right)(-\sin\alpha) + \left(\frac{y}{b}\right)(\cos\alpha) = 0$$

$$\Rightarrow \quad -\left(\frac{x}{a}\right)\sin\alpha + \left(\frac{y}{b}\right)\cos\alpha = 0 \qquad \text{... (2)}$$

Squaring and adding equations (1) and (2),

$$\left(\frac{x}{a}\right)^2\cos^2\alpha+\left(\frac{y}{b}\right)^2\sin^2\alpha+\frac{2xy}{ab}\sin\alpha\cos\alpha+\left(\frac{x}{a}\right)^2\sin^2\alpha+\left(\frac{y}{b}\right)^2\cos^2\alpha-\frac{2xy}{ab}\sin\alpha\cos\alpha=1+0$$

$$\Rightarrow \left(\frac{x}{a}\right)^2[\sin^2\alpha+\cos^2\alpha]+\left(\frac{y}{b}\right)^2[\sin^2\alpha+\cos^2\alpha]=1$$

$$\Rightarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1 \quad (\because \sin^2\alpha+\cos^2\alpha=1)$$

$\therefore \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ is the required envelope.

Q51. Find the envelope of the family of straight lines y = mx + $\sqrt{a^2m^2+b^2}$, m being a parameter.

Answer :

The given family of straight lines is,

$$y = mx + \sqrt{a^2m^2+b^2} \text{ zx} \quad \text{... (1)}$$

Differentiating partially with respect to 'm'

$$0 = 1.x + \frac{a^2(2m)}{2\sqrt{a^2m^2+b^2}}$$

$$-x = \frac{a^2m}{\sqrt{a^2m^2+b^2}}$$

$$\sqrt{a^2m^2+b^2} = \frac{a^2m}{-x}$$

Squaring on both sides, we get,

$$a^2m^2+b^2 = \frac{a^4m^2}{x^2}$$

$$\Rightarrow x^2a^2m^2+x^2b^2 = a^4m^2$$

$$\Rightarrow a^2m^2x^2-a^4m^2 = -b^2x^2$$

$$\Rightarrow a^2m^2(a^2-x^2) = x^2b^2$$

$$\therefore m^2 = \frac{b^2x^2}{a^2(a^2-x^2)} \quad \text{... (2)}$$

From equation (1),

$$y = mx + \sqrt{a^2m^2+b^2}$$

$$\Rightarrow y = mx - \frac{a^2m}{x}$$

$$\Rightarrow xy = mx^2 - a^2m$$

$$\Rightarrow xy = m(x^2-a^2)$$

Squaring on both sides,

$$x^2y^2 = m^2(x^2-a^2)^2$$

$$\Rightarrow \quad x^2y^2 = \frac{b^2x^2}{a^2(a^2-x^2)} \cdot (a^2-x^2)^2$$

$$\Rightarrow \quad y^2 = \frac{b^2}{a^2} \cdot (a^2-x^2)$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{1}{a^2} \cdot (a^2-x^2)$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{a^2}{a^2} - \frac{x^2}{a^2}$$

$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the envelope.

Q52. Show that the envelope of a circle whose centre lies on the parabola $y^2 = 4ax$ and which passes through its vertex is the cissoid: y^2 (2a +x) $x^3 = 0$.

Answer :

Given parabola is,

$$y^2 = 4ax \qquad \text{... (1)}$$

Let $P(at^2, 2at)$ be any point on the parabola and O (0, 0) be the vertex of the parabola.

Distance between OP = radius of the given family of circles.

$$\Rightarrow \sqrt{(at^2-0)^2 + (2at-0)^2} = r$$

$$\Rightarrow r = \sqrt{a^2t^4 + 4a^2t^2}$$

Equation of the circle is given by,

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\Rightarrow (x-at^2)^2 + (y-2at)^2 = \left(\sqrt{a^2t + 4a^2t^2}\right)^2$$

$$\Rightarrow x^2 - 2xat^2 + a^2t^4 + y^2 - 4yat + 4a^2t^2 = a^2t^2 + 4at^2$$

$$\Rightarrow x^2 + y^2 - 2at^2x - 4at\,y = 0 \qquad \text{... (2)}$$

Differentiating equation (2) partially with respect to 't',

$$-4at\,x - 4ay = 0$$

$$\Rightarrow 4at\,x = -4ay$$

$$\Rightarrow tx = -y$$

$$\therefore t = \frac{-y}{x}$$

Substituting the value of 't' in equation (2),

$$x^2 + y^2 - 2a\left(\frac{-y}{x}\right)^2 x - 4a\left(\frac{-y}{x}\right) y = 0$$

$$\Rightarrow x^2 + y^2 - \frac{2ay^2}{x} + \frac{4ay^2}{x} = 0$$

$$\Rightarrow x^3 + xy^2 + 2ay^2 = 0$$

$$\Rightarrow y^2(x+2a) + x^3 = 0 \quad \text{[Cissoid]}$$

$\therefore$ $y^2(x+2a) + x^3 = 0$ is required envelope.

Q53. Find the envelope of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the two parameters a, b are connected by the relation a + b = c, c, being a constant.

Answer :

The given family of ellipses is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{... (1)}$$

Where,

$$a + b = c \qquad \text{... (2)}$$

Differentiating equation (1) with respect to 't',

$$x^2\left(\frac{-2}{a^3}\right)\frac{da}{dt} + y^2\left(\frac{-2}{b^3}\right)\frac{db}{dt} = 0$$

$$\frac{x^2}{a^3}\left(\frac{da}{dt}\right) + \frac{y^2}{b^3}\left(\frac{db}{dt}\right) = 0$$

$$\frac{x^2}{a^3}\left(\frac{da}{dt}\right) = -\frac{y^2}{b^3}\left(\frac{db}{dt}\right) \qquad \text{... (3)}$$

Differentiating equation (2) with respect to 't',

$$\frac{da}{dt} + \frac{db}{dt} = 0$$

$$\frac{da}{dt} = -\frac{db}{dt} \qquad \text{... (4)}$$

Dividing equation (3) by equation (4) and equating it to $\frac{1}{c}$,

$$\frac{\frac{x^2}{a^3}\left(\frac{da}{dt}\right)}{\frac{da}{dt}} = \frac{\frac{-y^2}{b^3}\left(\frac{db}{dt}\right)}{-\frac{db}{dt}} = \frac{1}{c}$$

$$\Rightarrow \quad \frac{x^2}{a^3} = \frac{+y^2}{b^3} = \frac{1}{c}$$

$$\Rightarrow \quad \frac{x^2}{a^3} = \frac{1}{c} \quad , \quad \frac{y^2}{b^3} = \frac{1}{c}$$

$$\Rightarrow \quad cx^2 = a^3 \quad , \quad cy^2 = b^3$$

$$\Rightarrow \quad a = (cx^2)^{1/3} \quad , \quad b = (cy^2)^{1/3}$$

By substituting the values of a and b in equation (2),

$$c^{1/3}x^{2/3} + c^{1/3}y^{2/3} = c$$

$$x^{2/3} + y^{2/3} = c \times c^{-1/3} = c^{\frac{3-1}{3}}$$

$\therefore \quad x^{2/3} + y^{2/3} = c^{2/3}$, is the required envelope.

Q54. Determine envelope of one parameter family of curves of f(x, y, α) = 0 where α is the parameter. Also find the envelope of the straight lines x cosα + y sinα = l sinα cosα, α being parameter.

Answer :

Given family of straight lines is,

$x\cos\alpha + y\sin\alpha = l\sin\alpha\cos\alpha$

$$\Rightarrow \frac{x\cos a}{\sin\alpha\cos\alpha}+\frac{y\sin\alpha}{\sin\alpha\cos\alpha}=l$$

$$\Rightarrow \frac{x}{\sin\alpha}+\frac{y}{\cos\alpha}=l \quad \text{... (1)}$$

Differentiating equation (1) partially with respect to α,

$$\frac{x}{-\sin^2\alpha}\frac{d}{d\alpha}(\sin\alpha)+\frac{y}{-\cos^2\alpha}\left(\frac{d}{d\alpha}(\cos\alpha)\right)=0$$

$$\Rightarrow \frac{x}{-\sin^2\alpha}\cos\alpha+\frac{y}{-\cos^2\alpha}(-\sin\alpha)=0$$

$$\Rightarrow \frac{-x}{\sin^2\alpha}\cos\alpha+\frac{y\sin\alpha}{\cos^2\alpha}=0$$

$$\Rightarrow \frac{x\cos\alpha}{\sin^2\alpha}=\frac{y\sin\alpha}{\cos^2\alpha}$$

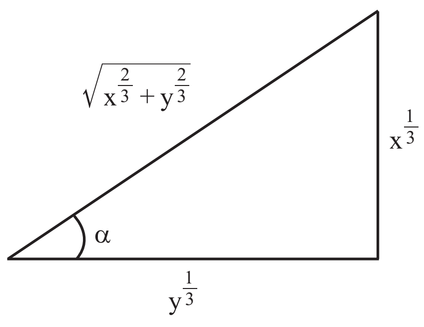
$$\Rightarrow x\cos^3\alpha = y\sin^3\alpha$$

$$\Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha}=\frac{x}{y}$$

$$\Rightarrow \tan^3\alpha=\frac{x}{y}$$

$$\Rightarrow \tan\alpha=\frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}$$

From the figure,



Figure

$$\sin\alpha=\frac{x^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}},\ \cos\alpha=\frac{y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}$$

Substituting the values of sinα, cosα in equation (1),

$$\frac{x}{\frac{x^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}}+\frac{y}{\frac{y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}}=l$$

$$\Rightarrow \frac{x\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}{x^{\frac{1}{3}}}+\frac{y\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}{y^{\frac{1}{3}}}=l$$

$$\Rightarrow x^{1-\frac{1}{3}}\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}+y^{1-\frac{1}{3}}\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}=l$$

$$\Rightarrow x^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}+y^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}=l$$

$$\Rightarrow \left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^{\frac{3}{2}}=l$$

$$\Rightarrow \left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=l^{\frac{2}{3}}$$

$\therefore$ $x^{\frac{2}{3}}+y^{\frac{2}{3}}=l^{\frac{2}{3}}$ is the envelope of straight line.

Q55. Find the envelope of the straight line xcost + ysint = a + a cost log tan $\frac{t}{2}$ where t is a parameter.

Answer :

Given equation is,

$$x\cos t + y\sin t = a + a\cos t\log\left(\tan\frac{t}{2}\right) \quad \text{... (1)}$$

Dividing equation (1) by cos(*t*),

$$x+y\tan t=\frac{a}{\cos t}+a\log\left(\tan\frac{t}{2}\right)$$

$$\Rightarrow x+y\tan t = a\sec t + a\log\left(\tan\left(\frac{t}{2}\right)\right) \quad \text{... (2)}$$

Differentiating equation (2) partially with respect to "*t*",

$$y\sec^2 t = a\sec t\tan t+\frac{a}{\tan\left(\frac{t}{2}\right)}\sec^2\frac{t}{2}\left(\frac{1}{2}\right)$$

$$= a\sec t\tan t+\frac{1}{2}\frac{a\cos\left(\frac{t}{2}\right)}{\sin\frac{t}{2}}\frac{1}{\cos^2\frac{t}{2}}$$

$$\Rightarrow y\sec^2 t = a\sec t\tan t+\frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$\Rightarrow y\sec^2 t = a\sec t\tan t+\frac{a}{\sin t}$$

$$\Rightarrow y=\frac{a\sec t\tan t}{\sec^2 t}+\frac{a}{\sin t\sec^2 t}$$

$$\Rightarrow y=\frac{a\left(\sec t.\frac{\sin^2 t}{\cos t}+a\right)}{\sin t\sec^2 t}$$

$$\Rightarrow y=\frac{a\left(\frac{\sin^2 t.\cos^2 t}{\cos^2 t}\right)}{\frac{\sin t}{\cos^2 t}}$$

$$\therefore y=\frac{a}{\sin t} \quad \text{... (3)}$$

Substituting equation (3) in equation (2),

$$x + \frac{a}{\sin t}\tan t = a\sec t + a\log\left(\tan\left(\frac{t}{2}\right)\right)$$

$$\Rightarrow \quad x + a\sec t = a\sec t + a\log\left(\tan\left(\frac{t}{2}\right)\right)$$

$$\Rightarrow \quad x = a\log\left(\tan\left(\frac{t}{2}\right)\right)$$

$$\Rightarrow \quad \frac{x}{a} = \log\left(\tan\left(\frac{t}{2}\right)\right)$$

$$\Rightarrow \quad e^{\frac{x}{a}} = \tan\frac{t}{2}$$

$$\Rightarrow \quad e^{\frac{-x}{a}} = \cot\frac{t}{2}$$

Since,

$$\cosh\left(\frac{x}{a}\right) = \frac{e^{\frac{x}{a}} + e^{\frac{-x}{a}}}{2}$$

$$\Rightarrow \quad 2\cosh\left(\frac{x}{a}\right) = \tan\frac{t}{2} + \cot\frac{t}{2}$$

$$= \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}$$

$$\Rightarrow \quad \cosh\left(\frac{x}{a}\right) = \frac{\sin^2\frac{t}{2} + \cos^2\frac{t}{2}}{2\cos\frac{t}{2}\sin\frac{t}{2}}$$

$$\Rightarrow \quad \cosh\left(\frac{x}{a}\right) = \frac{1}{\sin t} = \frac{y}{a} \quad [\because \text{From equation (3)}]$$

$$\Rightarrow \quad \cosh\left(\frac{x}{a}\right) = \frac{y}{a}$$

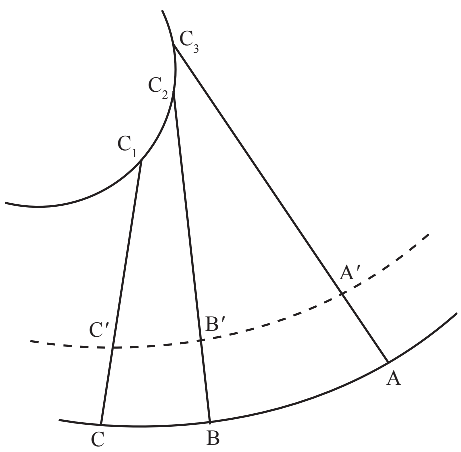
∴ The required equation is,

$$y = a\cosh\left(\frac{x}{a}\right).$$

2.5 EVOLUTES AND INVOLUTES

Q56. Prove that any curve have one evolute but an infinite number of involutes.

Answer :



Figure

Let ABC be the original curve, having C_1, C_2, C_3 as its evolute. Considering C_1, C_2, C_3 as the original curve, ABC is an involute of C_1, C_2, C_3. Draw a curve $A'B'C'$ of equal lengths of AA', BB' and CC', such that the curve $A'B'C'$ having same normal as the curve ABC. Therefore, having the same evolute. Hence, it is clear that any curve is given, it can have only one evolute, but an infinite number of curves may have the same evolute.

Any curve may have an infinite number of involutes.

Q57. Find the evolute of $y^2 = 4ax$.

Answer :

Given parabola is,

$$y^2 = 4ax \quad \text{... (1)}$$

Differentiating with respect to x,

$$2yy_1 = 4a$$

$$y_1 = \frac{2a}{y} \quad \text{... (2)}$$

Again differentiating with respect to x,

$$y_2 = \frac{-2a}{y^2}\, y_1 = \frac{-2a}{y^2}\left(\frac{2a}{y}\right) \quad (\because \text{From equation (2)})$$

$$y_2 = \frac{-4a^2}{y^3} \quad \text{... (3)}$$

$$1 + y_1^2 = 1 + \left[\frac{2a}{y}\right]^2$$

$$1 + y_1^2 = \frac{y^2 + 4a^2}{y^2} \quad \text{... (4)}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \quad ; \quad Y = y + \frac{1+y_1^2}{y_2}$$

$$X = x - \frac{\frac{2a}{y}\left[\frac{y^2+4a^2}{y^2}\right]}{\frac{-4a^2}{y^3}} \quad ; \quad Y = y + \frac{\frac{y^2+4a^2}{y^2}}{\frac{-4a^2}{y^3}}$$

$$X = x + \frac{y^3}{4a^2}\frac{2a}{y^3}(y^2 + 4a^2) \quad ; \quad Y = y - \frac{y^3}{4a^2}\frac{(y^2+4a^2)}{y^2}$$

$$X = x + \frac{1}{2a}(y^2 + 4a^2) \quad ; \quad Y = y - \frac{y}{4a^2}(y^2 + 4a^2)$$

$$\therefore X = 3x + 2a \quad [\because y^2 = 4ax] \; ; \quad Y = \frac{y}{4a^2}[4a^2 - y^2 - 4a^2]$$

$$; \quad Y = \frac{-[y^2][y]}{4a^2}$$

$$; \quad Y = \frac{-4ax\sqrt{4ax}}{4a^2}$$

$$; \quad Y = -\frac{2x^{3/2}\sqrt{a}}{a}$$

$$\therefore \quad x = 3x + 2a \quad \text{... (5)}$$

$$Y = \frac{-2x^{3/2}}{\sqrt{a}} \quad \text{... (6)}$$

The centre of curvature is,

$$(X, Y) = \left[3x + 2a, \ \frac{-2x^{3/2}}{\sqrt{a}}\right]$$

From equation (6), we have, $Y = \frac{-2x^{3/2}}{\sqrt{a}}$

Squaring on the both sides, equation (6),

$$Y^2 = \frac{4x^3}{a}$$

$$aY^2 = 4x^3$$

$$aY^2 = 4\left[\frac{X-2a}{3}\right]^3$$

∴ The equation of evolute is, $27ay^2 = 4(X–2a)^3$.

Q58. Show that the whole length of the evolute of the astroid x = cos³θ, y = a sin³θ is 12a.

Answer :

Given parametric equations of the astroid are,

$$x = a\cos^3\theta, \ y = a\sin^3\theta$$

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{3a\sin^2\cos\theta}{-3a\cos^2\theta\ \sin\theta}$$

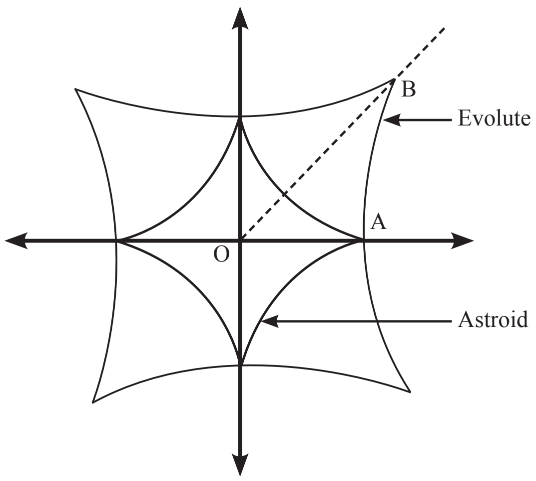
$$y_1 = -tan\theta$$

$$y_2 = \frac{d^2y}{dx^2}$$

$$= \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$$

$$= -\sec^2\theta \times \frac{1}{-3a\cos^2\theta.\sin\theta}$$

$$y_2 = \frac{1}{3a\cos^4\theta.\sin\theta}$$



Figure

The centre of curvature (x, y) is,

$$X = x - y_1\frac{(1+y_1^2)}{y_2}$$

$$= a\cos^3\theta + \tan\theta\,(1 + \tan^2\theta).3a\cos^4\theta\sin\theta$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta$$

Similarly,

$$Y = y + \frac{1+y_1^2}{y_2}$$

$$= a\sin^3\theta + 3a\cos^2\theta.\sin\theta$$

Consider,

$(x + y)^{2/3} + (x – y)^{2/3}$ (∵ To eliminate the parameter 't')

$$= a^{2/3}\,[(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2]$$

Therefore, the required evolute for the astroid is given as,

$$(X + Y)^{2/3} + (X - Y)^{2/3} = 2a^{2/3}$$

Hence the radius of curvature 'ρ' at any point of the given astroid is,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= (1 + \tan^2\theta)^{3/2}.3a\cos^4\theta.\sin\theta$$

$$\therefore\ \rho = \frac{3a}{2}\sin 2\theta$$

If the first quadrant, total length of the evolute (L), is 8 times the length of arc of the evoluate AB.

At the point A, radius of curvature of the astroid with $\theta = 0$ is given as,

$$\rho_1 = \frac{3a}{2}\sin 2(0) = 0$$

At the point B, radius of curvature with $\theta = \frac{\pi}{4}$ is given as,

$$\rho_2 = \frac{3a}{2}$$

The length of the arc of the evolute between two points is the difference between the radii of curvature of the curve at the corresponding points.

Thus, the length of the arc AB of the evolute

$$= \rho_2 - \rho_1$$

$$= \frac{3a}{2} - 0 = \frac{3a}{2}$$

Therefore,

∴ The whole length of the evolute

= 8 × length of the arc AB of the evolute

$$= 8\times\frac{3a}{2}$$

$$= 12a$$

Hence proved.

Q59. Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Answer :

Given that,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad x^2b^2 + y^2a^2 = a^2b^2 \quad \text{... (1)}$$

Differentiating equation (1) with respect to 'x', we get,

$$2xb^2 + 2y\frac{dy}{dx}a^2 = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx}a^2 = -2xb^2$$

$$\Rightarrow \quad y_1 = \frac{dy}{dx} = \frac{-xb^2}{ya^2} \quad \text{... (2)}$$

Differentiating again with respect to 'x', we get,

$$y_2 = \frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{y - x\left(\frac{-xb^2}{ya^2}\right)}{y^2}\right]$$

$$= \frac{-b^2}{a^2y^2}\times\left[y+\frac{x^2b^2}{ya^2}\right]$$

$$= \frac{-b^2}{a^2y^2}\times\frac{a^2y^2+x^2b^2}{ya^2}$$

$$= \frac{-b^2}{a^4y^3}(a^2b^2) \qquad [\because x^2b^2 + a^2y^2 = a^2b^2 \text{ from equation (1)}]$$

$$\therefore \quad y_2 = \frac{-b^4}{a^2y^3} \quad \text{... (3)}$$

The centre of curvature is given as,

$$X = x - \left[\frac{y_1(1+(y_1)^2)}{y_2}\right] \quad \text{... (4)}$$

$$Y = y + \left[\frac{1+(y_1)^2}{y_2}\right] \quad \text{... (5)}$$

Substituting y_1, y_2 values in equation (4),

$$\Rightarrow \quad X = x - \frac{\left(\frac{-xb^2}{ya^2}\right)\left(1+\left(\frac{-xb^2}{ya^2}\right)^2\right)}{\frac{-b^4}{a^2y^3}}$$

$$= x + \frac{xb^2}{ya^2}\left(1+\frac{x^2b^4}{y^2a^4}\right)\times\left(\frac{-a^2y^3}{b^4}\right)$$

$$= x - \frac{xy^2}{b^2}\left(\frac{y^2a^4 + x^2b^4}{y^2a^4}\right)$$

$$= x - \frac{x}{a^4b^2}(y^2a^4 + x^2b^4)$$

$$= \frac{xa^4b^2 - xy^2a^4 - x^3b^4}{a^4b^2}$$

$$= \frac{xa^2(a^2b^2) - xy^2a^4 - x^3b^4}{a^4b^2}$$

$$= \frac{xa^2(x^2b^2 + y^2a^2) - xy^2a^4 - x^3b^4}{a^4b^2} \quad [\because \text{From equation (1)}]$$

$$= \frac{x^3a^2b^2 + xa^4y^2 - xy^2a^4 - x^3b^4}{a^4b^2}$$

$$X = \frac{x^3b^2(a^2 - b^2)}{a^4b^2} = \frac{x^3(a^2 - b^2)}{a^4}$$

$$\Rightarrow \quad x^3(a^2 - b^2) = Xa^4$$

$$\Rightarrow \quad x^3 = \frac{Xa^4}{a^2 - b^2}$$

$$\Rightarrow \quad x = \left[\frac{Xa^4}{a^2 - b^2}\right]^{1/3}$$

$$x^2 = \left[\frac{Xa^4}{a^2 - b^2}\right]^{2/3}$$

Substituting y_1, y_2 values in equation (5),

$$Y = y + \left[\frac{1 + \left(\frac{-xb^2}{ya^2}\right)^2}{\frac{-b^4}{a^2y^3}}\right]$$

$$= y + \left[1 + \frac{x^2b^4}{y^2a^4}\right]\left[\frac{-a^2y^3}{b^4}\right]$$

$$= y + \frac{y^2a^4 + x^2b^4}{y^2a^4} \times \left(\frac{-a^2y^3}{b^4}\right)$$

$$= y - \left(\frac{y^2a^4 + x^2b^4}{a^2b^4}\right)y$$

$$= \frac{a^2yb^4 - y^3a^4 - x^2yb^4}{a^2b^4}$$

$$= \frac{y(a^2b^2)b^2 - y^3a^4 - x^2yb^4}{a^2b^4}$$

$$= \frac{yb^2(x^2b^2 + y^2a^2) - y^3a^4 - x^2yb^4}{a^2b^4} \quad [\because \text{From equation (1)}]$$

$$= \frac{x^2yb^4 + y^3b^2a^2 - y^3a^4 - x^2yb^4}{a^2b^4}$$

$$Y = \frac{y^3a^2(b^2 - a^2)}{a^2b^4} = \frac{y^3(b^2 - a^2)}{b^4}$$

$$X = \frac{x^3(a^2 - b^2)}{a^4}$$

$$Y = \frac{y^3(b^2 - a^2)}{b^4}$$

$y^3(b^2 - a^2) = Yb^4$

$$y = \left[\frac{Yb^4}{b^2 - a^2}\right]^{1/3}$$

$$y^2 = \left[\frac{Yb^4}{b^2 - a^2}\right]^{2/3}$$

Substituting x^2 and y^2 in the given ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

$$\left(\frac{1}{a^2}\right)\left[\frac{Xa^4}{a^2 - b^2}\right]^{2/3} + \left(\frac{1}{b^2}\right)\left[\frac{Yb^4}{b^2 - a^2}\right]^{2/3} = 1$$

$$\left(\frac{1}{a^2}\right)\left(\frac{Xa^{4\times 2/3}}{(a^2 - b^2)^{2/3}}\right) + \left(\frac{1}{b^2}\right)\left(\frac{Yb^{4\times 2/3}}{(b^2 - a^2)^{2/3}}\right) = 1$$

$$\frac{Xa^{\frac{8}{3}-2}}{(a^2 - b^2)^{2/3}} + \frac{Yb^{\frac{8}{3}-2}}{(b^2 - a^2)^{2/3}} = 1$$

$$\frac{Xa^{\frac{8-6}{3}}}{(a^2 - b^2)^{2/3}} + \frac{Yb^{\frac{8-6}{3}}}{(a^2 - b^2)^{2/3}} = 1 \qquad [\because (a^2 - b^2)^2 = (b^2 - a^2)^2]$$

$$\frac{Xa^{2/3} + Yb^{2/3}}{(a^2 - b^2)^{2/3}} = 1$$

$$\therefore Xa^{2/3} + Yb^{2/3} = (a^2 - b^2)^{2/3}$$

Q60. Show that the whole length of the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $4(a^2/b - b^2/a)$.

Answer :

Given ellipse equation is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{... (1)}$$

ρ at the point $(a \cos t, b \sin t)$ of equation (1) is equal to

$$\frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab}$$

The value of t is equal to 0 and $\frac{\pi}{2}$ at the ends of major and minor axes.

$\therefore \rho_1 = \rho$ at the end of major axis

$$= \frac{(b^2)^{3/2}}{ab}$$

$$= \frac{b^2}{a} \text{ and}$$

$\rho_2 = \rho$ at the end of minor axis.

$$= \frac{(a^2)^{3/2}}{ab}$$

$$= \frac{a^2}{b}$$

As the given ellipse is symmetrical about both the axes, its evolute must also be symmetrical about both the axes.

Therefore, the whole length of the evolute of the ellipse is,

$$4(\rho_2 - \rho_1) = 4\left(\frac{a^2}{b} - \frac{b^2}{a}\right)$$

Hence proved.

UNIT 3

MULTIVARIABLE CALCULUS (DIFFERENTIATION)

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Evaluate $\lim_{(x,y)\to(1,2)} \frac{x^2y}{x+y^2}$.

Answer : June/July-17, Q7

Given limit is,

$$\lim_{(x,y)\to(1,2)} \frac{x^2y}{x+y^2}$$

$$\Rightarrow \lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2y}{x+y^2} = \lim_{x\to 1}\left[\lim_{y\to 2} \frac{x^2y}{x+y^2}\right]$$

$$= \lim_{x\to 1}\left[\frac{x^2(2)}{x+(2)^2}\right]$$

$$= \lim_{x\to 1}\left[\frac{2x^2}{x+4}\right]$$

$$= \frac{2(1)^2}{1+4} = \frac{2}{5}$$

$$\therefore \lim_{(x,y)\to(1,2)} \frac{x^2y}{x+y^2} = \frac{2}{5}$$

Q2. Show that $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ doesn't exist.

Answer : Dec.-16, Q7

Given that,

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Let,

$$p_1 = \lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{xy^3}{x^2+y^6}\right]$$

$$= \lim_{y\to 0}\left[\underset{x\to 0}{lt} \frac{xy^3}{x^2+y^6}\right]$$

$$= \lim_{y\to 0}\left[\frac{0.y^3}{0+y^6}\right]$$

$$= 0$$

$$\therefore \quad p_1 = 0$$

Let,

$$p_2 = \lim_{\substack{y\to 0\\ x\to 0}}\left[\frac{xy^3}{x^2+y^6}\right]$$

$$= \lim_{x\to 0}\left[\underset{y\to 0}{lt}\frac{xy^3}{x^2+y^6}\right]$$

$$= \lim_{x\to 0}\left[\frac{x(0)}{x^2+0}\right]$$

$$= 0$$

$\therefore \quad p_2 = 0$

Since, $p_1 = p_2 = 0$, the limit doesn't exists.

Q3. Determine $\underset{(x,y)\to(0,0)}{Lt}\dfrac{xy^2}{x^2+y^4}$ if it exists.

Answer : Dec.-15, Q5

Given limit is,

$$\underset{(x,y)\to(0,0)}{Lim}\frac{xy^2}{x^2+y^4}$$

(i) $P_1 = \underset{\substack{x\to 0\\ y\to 0}}{Lim}\left[\dfrac{xy^2}{x^2+y^4}\right]$

$$= \underset{x\to 0}{Lim}\left[\underset{y\to 0}{Lim}\frac{xy^2}{x^2+y^4}\right]$$

$$= \underset{x\to 0}{Lim}\left[\frac{0}{x^2+0}\right]$$

$$= 0$$

$\therefore P_1 = 0$

(ii) $P_2 = \underset{\substack{x\to 0\\ y\to 0}}{Lim}\dfrac{xy^2}{x^2+y^4}$

$$= \underset{y\to 0}{Lim}\left[\underset{x\to 0}{Lim}\frac{xy^2}{x^2+y^4}\right]$$

$$= \underset{y\to 0}{Lim}\left[\frac{0}{0+y^4}\right]$$

$$= 0$$

(iii) Let, $y = mx$

$$P_3 = \underset{\substack{x\to 0\\ y\to mx}}{Lim}\frac{xy^2}{x^2+y^4} = \underset{x\to 0}{Lim}\frac{xm^2x^2}{x^2+m^4x^4} = \underset{x\to 0}{Lim}\frac{m^2x}{1+m^4x^2} = 0$$

(iv) Let, $y = mx^2$

$$P_4 = \underset{\substack{x\to 0\\ y\to mx^2}}{Lim}\frac{xy^2}{x^2+y^4} = \underset{x\to 0}{Lim}\frac{xm^2x^4}{x^2+m^4x^3} = \underset{x\to 0}{Lim}\frac{m^2x^3}{1+m^4x^5} = 0$$

$P_1 = P_2 = P_3 = P_4$

$$\therefore \quad \underset{\substack{x\to 0\\ y\to 0}}{Lim}\frac{xy^2}{x^2+y^4} = 0$$

Q4. Determine $\underset{(x,y)\to(1,-1)}{Lim} x^2 - y^2$.

Answer : April-16, Q5

Given limit is,

$$\underset{(x,y)\to(1,-1)}{Lim} x^2 - y^2$$

$$\Rightarrow \quad \underset{(x,y)\to(1,-1)}{Lim} = \underset{\substack{x\to 1\\ y\to -1}}{Lim} x^2 - y^2$$

$$= \underset{x\to 1}{Lim}\left[\underset{y\to -1}{Lim} x^2 - y^2\right]$$

$$= \underset{x\to 1}{Lim}\left[x^2 - (-1)^2\right]$$

$$= \underset{x\to 1}{Lim} x^2 - 1$$

$$= (1)^2 - 1$$

$$= 0$$

$$\therefore \quad \underset{(x,y)\to(1,-1)}{Lim} = 0$$

Q5. Show that $f(x, y) = \begin{cases}\dfrac{x-y}{x+y}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$ is discontinuous at the point (0, 0).

Answer : Dec.-12, Q7

The given function is,

$$f(x, y) = \begin{cases}\dfrac{x-y}{x+y}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$$

To check the continuity of the function $f(x, y)$, directly choose the path $y \to mx$.

$$\underset{\substack{y\to mx\\ x\to 0}}{Lt} f(x,y) = \underset{\substack{y\to mx\\ x\to 0}}{Lt}\left[\frac{x-y}{x+y}\right]$$

$$= \underset{x\to 0}{Lt}\left[\underset{y\to mx}{Lt}\frac{x-y}{x+y}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x-mx}{x+mx}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x(1-m)}{x(1+m)}\right]$$

$$= \underset{x\to 0}{Lt}\frac{1-m}{1+m}$$

i.e., the function depends on 'm' values only.

$\therefore$ The limit $x \to 0$ does not exists.

Hence, $f(x, y)$ is not continuous at (0, 0).

Q6. If x = r cos θ and y = r sin θ then find $\frac{\partial r}{\partial x}$.

Answer :

Given that,

$$x = r\cos\theta \quad \text{... (1)}$$

$$y = r\sin\theta \quad \text{... (2)}$$

Adding equations (1) and (2),

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2$$

$$x^2 + y^2 = r^2 \quad \text{... (3)}$$

Differentiating equation (3) partially with respect to 'x',

$$\Rightarrow \quad 2x = 2r\frac{\partial r}{\partial x}$$

$$\Rightarrow \quad x = r\frac{\partial r}{\partial x}$$

$$\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Q7. If u = $\frac{y}{z} + \frac{z}{x}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Answer :

Given equation is,

$$u = \frac{y}{z} + \frac{z}{x} \quad \text{... (1)}$$

In equation (1), u is dependent variable and x, y, z are independent variables.

(i) To Calculate $\frac{\partial u}{\partial x}$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{z} + \frac{z}{x}\right) = \left(0 + z\left(\frac{-1}{x^2}\right)\right)$$

[∵ y, z are constants]

$$= \frac{-z}{x^2} \quad \text{... (2)}$$

(ii) To Calculate $\frac{\partial u}{\partial y}$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{z} + \frac{z}{x}\right)$$

$$= \left(\frac{1}{z} + 0\right) \quad [\because x, z \text{ are constants}]$$

$$= \frac{1}{z} \quad \text{... (3)}$$

(iii) To Calculate $\frac{\partial u}{\partial z}$

$$\Rightarrow \quad \frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{z} + \frac{z}{x}\right) = \left(y\left(\frac{-1}{z^2}\right) + \frac{1}{x}\right)$$

[∵ x, y are constants]

$$= \frac{-y}{z^2} + \frac{1}{x} \quad \text{... (4)}$$

Consider,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(\frac{-z}{x^2}\right) + y\left(\frac{1}{z}\right) + z\left(\frac{-y}{z^2} + \frac{1}{x}\right)$$

$$= \frac{-z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$$

$$= 0$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Q8. If u = $\frac{x+y}{xy}$ find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Answer :

Given function is,

$$u = \frac{x+y}{xy}$$

$$\Rightarrow \quad u = \frac{x}{xy} + \frac{y}{xy}$$

$$\Rightarrow \quad u = \frac{1}{y} + \frac{1}{x} \quad \text{... (1)}$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial u}{\partial x} = -\frac{1}{x^2}$$

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial u}{\partial y} = -\frac{1}{y^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{1}{x^2} \text{ and } \frac{\partial u}{\partial y} = -\frac{1}{y^2}.$$

Q9. If $x^y + y^x = 1$, then find $\frac{dy}{dx}$.

Answer :

Given function is,

$$x^y + y^x = 1$$

Let, $f = x^y + y^x - 1 = 0$ ... (1)

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y - 0 \qquad \left[\because \frac{d}{dx} a^x = a^x \log a\right]$$

$$\Rightarrow \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiating equation (1), partially with respect to 'y',

$$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1} - 0$$

$$\Rightarrow \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$\frac{dy}{dx}$ is given as,

$$\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

Substituting the corresponding values in above equation,

$$\frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$$

$$\therefore \frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$$

Q10. Find $\frac{du}{dx}$ if u = sin(x² + y²), where a²x² + b²y² = c².

Answer :

Given that,

$$u = \sin(x^2 + y^2) \qquad \text{... (1)}$$

$$a^2x^2 + b^2y^2 = c^2 \qquad \text{... (2)}$$

Partially differentiating equation (1) with respect to 'x',

$$\frac{\partial u}{\partial x} = 2x\cos(x^2 + y^2)$$

$$\therefore \frac{\partial u}{\partial x} = 2x\cos(x^2 + y^2)$$

Partially differentiating equation (1) with respect to 'y',

$$\frac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$$

$$\therefore \frac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$$

Differentiating equation (2) with respect to 'x',

$$2a^2x + 2b^2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2b^2y\,dy}{dx} = -2a^2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-a^2x}{b^2y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \qquad [\because \text{From chain rule}] \qquad \text{... (3)}$$

Substituting the corresponding values in equation (3),

$$\frac{du}{dx} = 2x\cos(x^2 + y^2) + \left[2y\cos(x^2 + y^2)\left[\frac{-a^2x}{b^2y}\right]\right]$$

$$= 2x\cos(x^2 + y^2) + \frac{(-2xa^2\cos(x^2 + y^2))}{b^2}$$

$$= 2x\cos(x^2 + y^2)\left[\frac{b^2 - a^2}{b^2}\right]$$

$$\therefore \frac{du}{dx} = 2x\cos(x^2 + y^2)\left[\frac{b^2 - a^2}{b^2}\right]$$

Q11. If u = x³yeᶻ, where x = t, y = t² and z = logₑt, find $\frac{du}{dt}$ at t = 2.

Answer :

Given that,

$$u = x^3ye^z$$

$$x = t,\ y = t^2 \text{ and } z = \log_e t$$

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2t \text{ and } \frac{dz}{dt} = \frac{1}{t}$$

Differentiating 'u' partially with respect to x, y and z,

$$\frac{\partial u}{\partial x} = 3x^2ye^z \ ; \quad \frac{\partial u}{\partial y} = x^3e^z \ ; \quad \frac{\partial u}{\partial z} = x^3ye^z$$

From the definition of total derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 3x^2ye^z.1 + x^3e^z.2t + x^3ye^z.\frac{1}{t}$$

$$\frac{du}{dt} = 3x^2ye^z + x^3e^z2t + \frac{x^3y.e^z}{t}$$

Substituting x, y and z values in above equation,

$$\frac{du}{dt} = 3t^2.t^2e^{\log_e t} + t^3e^{\log_e t}.2t + \frac{t^3.t^2.e^{\log_e t}}{t}$$

$$= 3t^4e^{\log_e t} + 2t^4e^{\log_e t} + t^4e^{\log_e t}$$

$$= 3t^4.t + 2t^4.t + t^4.t \qquad [\because e^{\log_e t} = t]$$

$$= 3t^5 + 2t^5 + t^5$$

$$\therefore \frac{du}{dt} = 6t^5$$

For $t = 2$,

$$\therefore \frac{du}{dt} = 6(2)^5 = 192.$$

Verification by Direct Substitution

$$u = x^3ye^z$$

$$= t^3t^2e^{\log_e t}$$

$$u = t^5e^{\log_e t}$$

Differentiating above equation with respect to 't',

$$\frac{du}{dt} = t^5 + e^{\log_e t}\, 5t^4$$
$$= t^5 + 5t^5 = 6t^5$$

For $t = 2$,

$$\therefore \quad \frac{du}{dt} = 6(2)^5 = 192.$$

Q12. Find du/dt when u = x² y, x = t² and y = eᵗ.

Answer :

Given functions are,

$$u = x^2y \quad \text{... (1)}$$
$$x = t^2 \quad \text{... (2)}$$
$$y = e^t \quad \text{... (3)}$$

Differentiating equation (1) partially with respect to x and y,

$$\frac{\partial u}{\partial x} = 2xy, \quad \frac{\partial u}{\partial y} = x^2$$

Differentiating equation (2) partially with respect to t,

$$\frac{\partial x}{\partial t} = 2t$$

Differentiating equation (3) partially with respect to t,

$$\frac{\partial y}{\partial t} = e^t$$

From chain rule,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} \quad \text{... (4)}$$

Substituting the corresponding values in equation (4),

$$\frac{du}{dt} = (2xy)\,(2t) + (x^2)\,(e^t)$$
$$\Rightarrow \quad \frac{du}{dt} = 4xyt + x^2\, e^t \quad \text{... (5)}$$

Substituting equations (2) and (3) in equation (5),

$$\frac{du}{dt} = 4(t^2)\,(e^t)t + (t^2)^2e^t$$
$$= 4t^3e^t + t^4e^t$$
$$= t^3e^t\,(4 + t)$$
$$\therefore \quad \frac{du}{dt} = t^3e^t\,(4 + t).$$

Q13. Find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ if u = f(x + y, x – y).

Answer :

Given that,

$$u = f\,(x + y, x - y)$$

Let, $a = x + y$ and

$$b = x - y$$

Then, $u = f(a, b)$

$$\frac{\partial a}{\partial x} = 1, \quad \frac{\partial a}{\partial y} = 1$$
$$\frac{\partial b}{\partial x} = 1, \quad \frac{\partial b}{\partial y} = -1$$

Applying chain rule of partial differentiation,

$$\text{i.e.,} \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x}$$
$$= \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(1)$$
$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \quad \text{... (1)}$$

Similary,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y}$$
$$= \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(-1)$$
$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial b} \quad \text{... (2)}$$

Adding equations (1) and (2),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} + \frac{\partial u}{\partial a} - \frac{\partial u}{\partial b}$$
$$= 2\frac{\partial u}{\partial a}$$
$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial a}$$

Q14. Find $\frac{du}{dt}$ when u = x² + y², x = at², y = 2at.

Answer :

Given functions are,

$$u = x^2 + y^2 \quad \text{... (1)}$$
$$x = at^2 \quad \text{... (2)}$$
$$y = 2at \quad \text{... (3)}$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial u}{\partial x} = 2x$$

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial u}{\partial y} = 2y$$

Differentiating equation (2) partially with respect to 't',

$$\frac{\partial x}{\partial t} = a\,(2t) = 2at$$

Differentiating equation (3) partially with respect to 't',

$$\frac{\partial y}{\partial t} = 2a$$

From chain rule,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\,\frac{\partial y}{\partial t} \quad \text{... (4)}$$

Substituting the corresponding values in equation (4),

$$= 2x(2at) + 2y(2a)$$

$$= 4xat + 4ay$$

$$\therefore \quad \frac{du}{dt} = 4xat + 4ay \quad \text{... (5)}$$

Substituting equations (2) and (3) in equation (5),

$$\frac{du}{dt} = 4(at^2)at + 4a(2at)$$

$$= 4a^2t^3 + 8a^2t$$

$$= 4at\,(t^2 + 2)$$

$$\therefore \quad \frac{du}{dt} = 4at\,(t^2 + 2).$$

Q15. If u = f(y – z, z – x, x – y), find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

Answer :

Given that,

$$u = f(y - z, z - x, x - y)$$

Let,

$$a = y - z$$
$$b = z - x$$
$$c = x - y$$

Then,

$$u = f(a, b, c)$$
$$a = y - z; \quad b = z - x; \quad c = x - y$$

$$\frac{\partial a}{\partial x} = 0; \quad \frac{\partial b}{\partial x} = -1; \quad \frac{\partial c}{\partial x} = 1$$

$$\frac{\partial a}{\partial y} = 1; \quad \frac{\partial b}{\partial y} = 0; \quad \frac{\partial c}{\partial y} = -1$$

$$\frac{\partial a}{\partial z} = -1; \quad \frac{\partial b}{\partial z} = 1; \quad \frac{\partial c}{\partial z} = 0$$

Applying chain rule of partial differentiation.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x}$$

$$= \frac{\partial u}{\partial a}(0) + \frac{\partial u}{\partial b}(-1) + \frac{\partial u}{\partial c}(1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial c} - \frac{\partial u}{\partial b} \quad \text{... (1)}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial y}$$

$$= \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(0) + \frac{\partial u}{\partial c}(-1)$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} \quad \text{... (2)}$$

And $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z}$$

$$= \frac{\partial u}{\partial a}(-1) + \frac{\partial u}{\partial b}(1) + \frac{\partial u}{\partial c}(0)$$

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial b} - \frac{\partial u}{\partial a} \quad \text{... (3)}$$

Adding equations (1), (2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial c} - \frac{\partial u}{\partial b} + \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} - \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 .$$

Q16. Define the Jacobian of the functions $u_1, u_2, ..., u_n$ of n variables $x_1, x_2, ..., x_n$.

Answer :

Definition

If $u_1, u_2, ..., u_n$ be the functions of 'n' variables $x_1, x_2, ..., x_n$, then the determinant,

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, ..., u_n$ with respect to $x_1, x_2, ..., x_n$ and is denoted by $J(u_1, u_2, ..., u_n)$ or $\frac{\partial(u_1, u_2, ... u_n)}{\partial(x_1, x_2, ... x_n)}$

Q17. If x = uv, y = $\frac{u}{v}$ then find $\frac{\partial(x, y)}{\partial(u, v)}$

Answer :

Given functions are,

$$x = uv \quad \text{... (1)}$$

$$y = \frac{u}{v} \quad \text{... (2)}$$

Differentiating the equation (1) with respect to u, v,

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u$$

Differentiating equation (2) with respect to u, v,

$$\frac{\partial y}{\partial u} = \frac{1}{v}, \quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Substituting the corresponding values in above equation,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$= v\left(\frac{-u}{v^2}\right) - u\left(\frac{1}{v}\right)$$

$$= -\frac{uv}{v^2} - \frac{u}{v}$$

$$= \frac{-u}{v} - \frac{u}{v}$$

$$= \frac{-2u}{v}$$

$$\therefore \frac{\partial(x,y)}{\partial(u.v)} = \frac{-2u}{v}$$

Q18. If x = u(1 + v) and y = v(1 + u), find ∂(x, y) / ∂ (u, v).

Answer :

Given functions are,

$x = u\,(1+v)$ ... (1)

$y = v\,(1+u)$ ... (2)

Differentiating equation (1) partially with respect to u and v,

$$\frac{\partial x}{\partial u} = 1 + v,\ \frac{\partial x}{\partial v} = u$$

Differentiating equation (2) partially with respect to u and v,

$$\frac{\partial y}{\partial u} = v,\ \frac{\partial y}{\partial v} = 1 + u$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad ...(3)$$

Substituting the corresponding values in equation (3),

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + v + u + uv - uv$$

$$= 1 + u + v$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = 1 + u + v.$$

Q19. If x = r cosθ, y = r sinθ, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

OR

Find the Jacobian of the transformation x = r cosθ, y = r sinθ.

OR

Find $\frac{\partial(x,y)}{\partial(r,\theta)}$, if x = r cos θ, y = r sin θ.

Answer :

Given that,

$x = r\cos\theta$ ... (1)

$y = r\sin\theta$ ... (2)

Differentiating equations (1) and (2) partially with respect to 'r'.

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial y}{\partial r} = \sin\theta$$

Differentiating equations (1) and (2) partially with respect to 'θ',

$$\frac{\partial x}{\partial \theta} = -r\sin\theta\ ;\ \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \cos\theta\,(r\cos\theta) - (-r\sin\theta)\sin\theta$$

$$= r(\sin^2\theta + \cos^2\theta)$$

$$= r$$

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

Q20. If x = r cos θ, y = r sinθ, find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

OR

If x = r cos θ, y = r sinθ, then find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Answer :

Given that,

$x = r\cos\theta$ ... (1)

$y = r\sin\theta$ ... (2)

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}}$$

[∵ From the properties of Jacobians]

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}} \quad ...(3)$$

Differentiating equations (1) and (2) partially with respect to 'r'

$$\frac{\partial x}{\partial r} = \cos\theta \; ; \; \frac{\partial y}{\partial r} = \sin\theta$$

Differentiating equations (1) and (2) partially with respect to 'θ'

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \; ; \; \frac{\partial y}{\partial \theta} = r\cos\theta$$

Substituting the corresponding values in equation (3),

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}}$$

$$= \frac{1}{r\cos\theta(\cos\theta) - \sin\theta(-r\sin\theta)}$$

$$= \frac{1}{r\cos^2\theta + r\sin^2\theta}$$

$$= \frac{1}{r(\cos^2\theta + \sin^2\theta)}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}.$$

Q21. If x = u(1 – v), y = uv, show that JJ' = 1.

Answer :

Given that,

$x = u(1-v), y = uv$

$\Rightarrow \quad x = u - uv = u - y$

$\Rightarrow \quad u = x + y$

$\Rightarrow \quad y = (x+y)v$

$\Rightarrow \quad v = \dfrac{y}{x+y}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = u - uv \Rightarrow \frac{\partial x}{\partial u} = 1 - v, \; \frac{\partial x}{\partial v} = -u$$

$$y = uv \Rightarrow \frac{\partial y}{\partial u} = v, \; \frac{\partial y}{\partial v} = u$$

$$\therefore \quad J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$u = x + y$

$$y = uv \Rightarrow v = \frac{y}{u} = \frac{y}{x+y}$$

$$\frac{\partial u}{\partial x} = 1, \; \frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y)-y}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} - \frac{-y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore \ JJ' = u.\frac{1}{u} = 1.$$

Q22. Write the expressions for Taylor's series for function of two variables.

Answer :

The expression for Taylor's series for function of two variables is,

$$f(x,y) = f(a,b) + \left[\frac{(x-a)}{1!} f_x(a,b) + \frac{(y-b)}{1!} f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right] + ...$$

Q23. Find the Taylor series expansion of x^y near the point (1, 1) upto the first degree terms.

Answer :

Given function is,

$f(x, y) = x^y$

$f(1, 1) = 1^{(1)} = 1$

$f_x(x, y) = yx^{y-1}$; $f_x(1, 1) = 1(1)^{1-1} = 1$

$f_y(x, y) = x^y \log x$; $f_y(1, 1) = 0$

From Taylor's series,

$\Rightarrow \ f(x, y) = f(1, 1) + (x - 1) f_x(1, 1) + (y - 1) f_y(1, 1)$

$x^y = 1 + (x - 1)(1) + (y - 1)0$

$= 1 + x - 1 + 0$

$\therefore \ x^y = x.$

Q24. Define maximum and minimum values for a function of two variables.

Answer :

Maximum Value

A function $f(x, y)$ is said to have maximum value at $x = a$ and $y = b$ if and only if,

$f(a, b) > f(a + h, b + k)$

Where,

h, k are small values.

Minimum Value

A function $f(x, y)$ is said to have minimum value at $x = a$ and $y = b$ if and only if,

$f(a, b) < f(a + h, b + k)$

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

3.1 FUNCTIONS OF TWO VARIABLES, LIMITS AND CONTINUITY

Q25. Explain function of several variables.

Answer :

Function of Several Variables

A function which contains more than one variable is called function of several variables.

(i) Function of Two Variables: A function which contains two variables is called function of two variables.

It is of the form,

$z = f(x,y)$

Example: $z = x^2 + y^2$

(ii) Function of Three Variables: A function which has three variables is called function of three variables.

It is of the form,

$v = f(x, y, z)$

Example: $v = x^2 + y^2 + z^2$.

Q26. Define limit and continuity of a function. Also mention the steps involved in obtaining the limit of a function of two variables.

Answer :

Limit

A function $f(x, y)$ is said to have limit L as 'x' tends to 'a' and 'y' tends to 'b' i.e,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If for every positive number $\varepsilon > 0$, There exists a corresponding number $\delta > 0$, such that,

$$|f(x, y) - L| < \varepsilon \text{ when } |x - a| < \delta \text{ and } |y - b| < \delta$$

Continuity

A function $f(x, y)$ at a point (a, b) is said to be continuous, if it is continuous at each point such that,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a, b)$$

Procedure

Step 1

Initially, determine the lim $f(x, y)$ along path p_1

i.e, $x \to a$ and $x \to b$

Step 2

In the next step, determine lm $f(x, y)$ along path p_2

i.e, $y \to b$ and $x \to a$

Step 3

In this step obtain the limit along path $y = mx$ or $y = mx^n$, if the value of a and b are 0.

Q27. If $f(x, y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$**, show that** $\dfrac{\partial^2 f}{\partial x \partial y} \neq \dfrac{\partial^2 f}{\partial y \partial x}$ **at (0, 0).**

Answer : June/July-17, Q14(a)

Given that,

$$f(x, y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

$$f(x, y) = \frac{x^2 y(x-y)}{x^2+y^2}$$

$$f(0, 0) = 0 \quad \text{... (1)}$$

$$\Rightarrow f(\Delta x, 0) = \frac{(\Delta x)^2(0)(\Delta x - 0)}{(\Delta x)^2 + (0)^2}$$

$$= \frac{(0)}{(\Delta x)^2}$$

$$f(\Delta x, 0) = f(x, 0) = 0 \quad \text{... (2)}$$

$$\Rightarrow f(0, \Delta y) = \frac{(0)^2 \Delta y(0 - \Delta y)}{(0)^2(\Delta y)^2}$$

$$= \frac{0}{(\Delta y)^2}$$

$$\therefore \quad f(0, \Delta y) = f(0, y) = 0 \quad \text{... (3)}$$

Since,

$$\frac{\partial f}{\partial x} = f_x(x, y)$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \frac{0-0}{\Delta x} \quad \text{[From equation (1) and (2)]}$$

$$= 0$$

$$\therefore f_x(0, 0) = 0 \quad \text{... (4)}$$

$$f_x(0, y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^2 y(\Delta x - y)}{(\Delta x)^2 + y^2} - 0}{\Delta x}$$

[$\because$ From equation (3)]

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 y(\Delta x - y)}{\Delta x((\Delta x)^2 + y^2)}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x) y(\Delta x - y)}{(\Delta x)^2 + y^2}$$

$$= \frac{(0) y(0-y)}{(0)^2 + y^2} = \frac{0(0-y)}{y^2}$$

$$= \frac{0}{y^2} = 0$$

$$\Rightarrow f_x(0, \Delta y) = 0$$

$$\therefore f_x(0, \Delta y) = 0 \quad \text{... (5)}$$

Similarly,

$$\frac{\partial f}{\partial y} = f_y(x, y)$$

$$= \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0-0}{\Delta y}$$

[From equations (1) and (2)]

$$= 0$$

$$\therefore f_y(0, 0) = 0 \quad \text{... (6)}$$

$$f_y(x, 0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\frac{x^3(\Delta y)(x - \Delta y)}{x^2 + (\Delta y)^2} - 0}{\Delta y}$$

[From equation (3)]

$$= \lim_{\Delta y \to 0} \frac{x^3(\Delta y)(x - \Delta y)}{\Delta y(x^2 + (\Delta y)^2)}$$

$$= \lim_{\Delta y \to 0} \frac{x^3(x - \Delta y)}{x^2 + (\Delta y)^2}$$

$$= \frac{x^3(x-0)}{x^2 + (0)^2}$$

$$= \frac{x^4}{x^2} = x^2$$

$$\Rightarrow f_y(x, 0) = x^2 = 0$$

$$\therefore f_y(\Delta x, 0) = (\Delta x)^2 \quad \text{... (7)}$$

Now,

$$\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{(0,0)} = f_{xy}(0, 0)$$

$$= \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 - 0}{\Delta x}$$

[From equation (6) and (7)]

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (\Delta x) = 0$$

$$\therefore \left[\frac{\partial^2 f}{\partial x \partial y}\right]_{(0,0)} = 0$$

And,

$$\left[\frac{\partial^2 f}{\partial y \partial x}\right]_{(0,0)} = f_{yx}(0, 0)$$

$$= \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0-0}{\Delta y}$$

[From equations (4) and (5)]

$$= \lim_{\Delta y \to 0} \frac{0}{\Delta y}$$

$$= \frac{0}{0} = 0$$

$$\therefore \left[\frac{\partial^2 f}{\partial y \partial x}\right]_{(0,0)} = 0$$

$$\therefore \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 \text{ at } (0, 0)$$

Hence proved.

Q28. Show that the function

$$f(x,y)=\begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$ **is not differentiable**

at (0, 0).

Answer : **Dec.-16, Q14(a)**

Given function is,

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$\Rightarrow \quad f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

$$f(0, 0) = 0$$

$$f(\Delta x, 0) = \frac{(\Delta x)^2 - 0^2}{(\Delta x)^2 + 0^2} = 1$$

$$\Rightarrow \quad f(\Delta x, 0) = 1$$

$$\Rightarrow \quad f(\Delta x, 0) = f(x, 0) = 1$$

$$f(0, \Delta y) = \frac{0^2 - (\Delta y)^2}{0^2 + (\Delta y)^2}$$

$$\Rightarrow \quad f(0, \Delta y) = -1$$

$$\Rightarrow \quad f(0, \Delta y) = f(0, y) = -1$$

(i) $$\frac{\partial f}{\partial x} = f_x(x, y) = \underset{\Delta x \to 0}{Lim} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\Rightarrow \quad f_x(0,0) = \underset{\Delta x \to 0}{Lim} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lim} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lim} \frac{1-0}{\Delta x} = \underset{\Delta x \to 0}{Lim} \frac{1}{\Delta x} = \infty$$

$$\therefore \quad f_x(0, 0) = \infty$$

(ii) $$\frac{\partial f}{\partial y} = f_y(x, y) = \underset{\Delta y \to 0}{Lim} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$\Rightarrow \quad f_y(0, 0) = \underset{\Delta y \to 0}{Lim} \frac{f\ (0, \Delta y) - (0, 0)}{\Delta y}$$

$$= \underset{\Delta y \to 0}{Lim} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y}$$

$$= \underset{\Delta y \to 0}{Lim} \frac{-1-0}{\Delta y} = \underset{\Delta y \to 0}{Lim} \frac{-1}{\Delta y} = -\infty$$

$$\therefore \quad f_y(0, 0) = -\infty$$

$$f_x(0, 0) \neq f_y(0, 0)$$

$\therefore$ f is not differentiable.

Q29. Show that the function $$f(x,y)=\begin{cases} \dfrac{x^2+y^2}{x-y} & (x,y) \neq (0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

is not continuous at (0,0).

June-13, Q14(b)

OR

Show that $$f(x, y) = \begin{cases} \dfrac{x^2+y^2}{x-y} & (x,y) \neq (0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$ **is not**

continuous at (0, 0).

Answer : **June-11, Q12(b)**

Given that,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x - y}$$

Let,

$$P_1 = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^2 + y^2}{x - y}\right]$$

$$= \lim_{y \to 0} \left[\lim_{x \to 0} \frac{x^2 + y^2}{x - y}\right] = \lim_{y \to 0} \left[\frac{0 + y^2}{0 - y}\right]$$

$$= \lim_{y \to 0} \left[-\frac{y^2}{y}\right] = \lim_{y \to 0} [-y]$$

$$= \lim_{y \to 0} [-0] = 0$$

$$\therefore \quad P_1 = 0 \qquad \dots (1)$$

Let,

$$P_2 = \lim_{\substack{y\to 0\\x\to 0}} \frac{x^2+y^2}{x-y}$$

$$= \lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+0}{x-0}\right] = \lim_{x\to 0}\left[\frac{x^2}{x}\right]$$

$$= \lim_{x\to 0}[x] = 0$$

$$\therefore \quad P_2 = 0 \qquad \text{... (2)}$$

Let,

$$P_3 = \lim_{\substack{y\to mx\\x\to 0}}\left[\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\lim_{y\to mx}\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+(mx)^2}{x-mx}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+m^2x^2}{x(1-m)}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2(1+m^2)}{x(1-m)}\right]$$

$$= \lim_{x\to 0}\left[\frac{x(1+m^2)}{(1-m)}\right] = \frac{0(1+m^2)}{(1-m)} = 0$$

$$\therefore \quad P_3 = 0 \qquad \text{... (3)}$$

Let,

$$P_4 = \lim_{\substack{y\to mx^2\\x\to 0}}\left[\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\lim_{y\to mx^2}\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+(mx^2)^2}{x-mx^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+m^2x^4}{x-mx^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2(1+m^2x^2)}{x(1-mx)}\right]$$

$$= \lim_{x\to 0}\left[\frac{x(1+m^2x^2)}{(1-mx)}\right]$$

$$= \frac{0[1+m^2(0)^2]}{[1-m(0)]} = 0$$

$$\therefore \quad P_4 = 0 \qquad \text{... (4)}$$

Since the value of limit along any path is same, the limit exists and the limit value is zero.

Therefore, the $f(x,y)$ is continuous at origin only if $(x,y) = 0$ for $x = 0$ and $y = 0$ otherwise the function $f(x,y)$ is not continuous at (0,0).

3.2 PARTIAL DERIVATIVES

Q30. Explain the concept of partial derivatives.

Answer :

Partial derivatives

Consider a function, $u = f(x, y)$

Where, x, y – Two independent variables.

The derivative of function 'u' with respect to 'x' by keeping another variable 'y' as constant is known as partial derivative of 'u' with respect to y.

The partial derivative of a function with respect to x is denoted by $\frac{\partial u}{\partial x}$

Similarly, partial derivative of a function with respect to 'y' is denoted, by $\frac{\partial u}{\partial y}$

Second Order Partial Derivative

The derivative of a first order partial derivative $\left(i.e., \frac{\partial u}{\partial x}\right)$ is called second order partial derivative.

The second order partial derivative of $\frac{\partial u}{\partial x}$ with respect to 'x' is,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

Where, $u = f(x, y)$

Similarly, the second order partial derivative of $\frac{\partial u}{\partial y}$ with respect to 'y' is ,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

Differentiating a function $\frac{\partial u}{\partial x}$ with respect to 'y',

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

Differentiating a function $\frac{\partial u}{\partial y}$ with respect to 'x',

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

Q31. If x = r cos θ, y = r sin θ, find $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$

Answer :

Given equations are,

$x = r\cos\theta$

$y = r\sin\theta$

Squaring and adding the above equations,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$\Rightarrow \quad x^2 + y^2 = r^2$$

$$\Rightarrow \quad r = \sqrt{x^2+y^2} \qquad \text{... (1)}$$

Differentiating equation (1) partially with respect to x ,

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$$

$$= \frac{x}{\sqrt{x^2+y^2}} \qquad \text{... (2)}$$

Differentiating equation (1) partially with respect to y,

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \times 2y$$

$$= \frac{y}{\sqrt{x^2+y^2}} \qquad \text{... (3)}$$

Squaring and adding equations (2) and (3),

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \left[\frac{x}{\sqrt{x^2+y^2}}\right]^2 + \left[\frac{y}{\sqrt{x^2+y^2}}\right]^2$$

$$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2}$$

$$= 1$$

$$\therefore \quad \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1\,.$$

Q32. If z = e^ax+by f(ax – by), then prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

Answer :

Given that,

$$z = e^{ax+by}\cdot f(ax-by)$$

Consider, $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$ ...(1)

Consider,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[e^{ax+by}\cdot f(ax-by)]$$

$$= e^{ax+by} \times f'(ax-by)(a) + f(ax-by) \times e^{ax+by}(a)$$

$$\therefore \quad \frac{\partial z}{\partial x} = ae^{ax+by}[f'(ax-by) + f(ax-by)] \qquad \text{... (2)}$$

Consider, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[e^{ax+by} \cdot f(ax-by)]$

$= e^{ax+by} \times f'(ax-by) \times (-b) + f(ax-by) \times e^{ax+by}(b)$

$\therefore \quad \frac{\partial z}{\partial y} = b.e^{ax+by}[-f'(ax-by) + f(ax-by)]$

Substituting equations (2) and (3) in equation (1), ... (3)

$\therefore \quad b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = bae^{ax+by}[f'(ax-by) + f(ax-by)] + abe^{ax+by}[-f'(ax-by) + f(ax-by)]$

$= abe^{ax+by}[f'(ax-by) + f(ax-by) - f'(ax-by) + f(ax-by)]$

$= abe^{ax+by}[2.f(ax-by)]$

$= 2abz \qquad [\because z = e^{ax+by}f(ax-by)]$

$\therefore \quad b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$

Q33. If u = $(x^2+y^2+z^2)^{\frac{-1}{2}}$ then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Answer :

Given that,

$u = (x^2+y^2+z^2)^{-1/2}$

Differentiating equation (1) partially with respect to x.

$\frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1}2x$

$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2x$

$\Rightarrow \quad \frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$

$\frac{\partial u^2}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]$

$= \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$

$= -\left[x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{\frac{-3}{2}-1}(2x) + (x^2+y^2+z^2)^{\frac{-3}{2}}\right]$

$= 3x^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$ ... (2)

Similarly,

$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$ ... (3)

And,

$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$ ... (4)

Adding equations (2), (3) and (4),

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \left[3x^2(x^2+y^2+z^2)^{\frac{-5}{2}}+(x^2+y^2+z^2)^{\frac{-3}{2}}\right]+\left[3y^2(x^2+y^2+z^2)^{\frac{-5}{2}}+(x^2+y^2+z^2)^{\frac{-3}{2}}\right]$$

$$+\left[3z^2(x^2+y^2+z^2)^{\frac{-5}{2}}+(x^2+y^2+z^2)^{\frac{-3}{2}}\right]$$

$$= 3(x^2+y^2+z^2)^{\frac{-5}{2}}[x^2+y^2+z^2]+3(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$= 3(x^2+y^2+z^2)^{\frac{-3}{2}}[(x^2+y^2+z^2)^{-1}(x^2+y^2+z^2)+1]$$

$$= 3(x^2+y^2+z^2)^{\frac{-3}{2}}[1+1]$$

$$= 3(x^2+y^2+z^2)^{\frac{-3}{2}}(2)$$

$$= 6(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 6(x^2+y^2+z^2)^{\frac{-3}{2}}$$

3.3 TOTAL DIFFERENTIAL AND DIFFERENTIABILITY

Q34. Write a brief note on total derivatives.

Answer :

Let, $u = f(x, y, z)$, where x, y, z are function of a one independent variable 't' i.e., $x = x(t), y = y(t), z = z(t)$. Then considering the dependent variable 'f' as a function of single independent variable 't'. So, the derivative of function 'f' with respect to 't' is called as total derivative of 'f' which is given as,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt} \quad \text{... (1)}$$

If y and z are functions of x, then f is a function of single independent variable 'x'. Hence, s equation (1) becomes,

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}$$

$$\therefore \quad \frac{df}{dx} = \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}$$

The above equation is called as chain rule of partial differentiation for one independent variable. This rule can be extented to functions of more than two independent variables.

Q35. Find $\frac{du}{dt}$ if $u = \tan^{-1}\left(\frac{y}{x}\right)$ and $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$.

Answer :

Given that,

$$u = \tan^{-1}\left(\frac{y}{x}\right) \text{ and}$$

$$x = e^t - e^{-t} \; ; \; y = e^t + e^{-t}$$

Differentiating above equations above equations with respect to 't',

$$\frac{dx}{dt} = e^t + e^{-t} \; ; \; \frac{dy}{dt} = e^t - e^{-t}$$

Differentiating 'u' partially with respect to x and y,

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{-x^2 y}{x^2+y^2}\cdot\frac{1}{x^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{-y}{(x^2+y^2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$$

From the definition of total derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Substituting the corresponding values in above equation,

$$\frac{du}{dt} = \frac{-y}{x^2+y^2}(e^t+e^{-t}) + \frac{x}{x^2+y^2}(e^t-e^{-t})$$

Substituting 'x' and 'y' values in above equation,

$$\frac{du}{dt} = \frac{-(e^t+e^{-t})(e^t+e^{-t})}{(e^t-e^{-t})^2+(e^t+e^{-t})^2} + \frac{(e^t-e^{-t})(e^t-e^{-t})}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}$$

$$= \frac{1}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}\,[-(e^t+e^{-t})(e^t+e^{-t}) + (e^t-e^{-t})(e^t-e^{-t})]$$

$$= \frac{1}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}\,[-(e^t+e^{-t})^2 + (e^t-e^{-t})^2]$$

$$= \frac{-(e^{2t}+e^{-2t}+2)+(e^{2t}+e^{-2t}-2)}{e^{2t}+e^{-2t}-2+e^{2t}+e^{-2t}+2}$$

$$= \frac{-e^{2t}-e^{-2t}-2+e^{2t}+e^{-2t}-2}{2e^{2t}+2e^{-2t}}$$

$$= \frac{-4}{2(e^{2t}+e^{-2t})}$$

$$= \frac{-2}{e^{2t}+e^{-2t}}$$

$$= \frac{-2}{2\cosh 2t}$$

$$= \frac{-1}{\cosh 2t}$$

$$\therefore \quad \frac{du}{dt} = \frac{-1}{\cosh 2t}.$$

Q36. Find $\frac{du}{dt}$ as a total derivative if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$.

Answer :

Given that,

$u = x^2 + y^2 + z^2$

$x = e^{2t}$; $y = e^{2t}\cos 3t$; $z = e^{2t}\sin 3t$

Differentiating above equations with respect to t,

$$\frac{dx}{dt} = 2e^{2t} \; ; \; \frac{dy}{dt} = 2\,e^{2t}\cos 3t - 3e^{2t}\sin 3t$$

$$\frac{dz}{dt} = 2\,e^{2t}\sin 3t + 3e^{2t}\cos 3t$$

Differentiating u partially with respect to x and y,

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial z} = 2z$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Substituting the corresponding values in above equation,

$$\frac{du}{dt} = 2x\,(2e^{2t}) + 2y\,[e^{2t}\,(2\cos 3t - 3\sin 3t)] + 2z\,[e^{2t}\,(2\sin 3t + 3\cos 3t)]$$

Substituting the values of x, y and z in above equation,

$$\frac{du}{dt} = 4\,e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\cos 3t\sin 3t + 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t\cos 3t$$

$$\Rightarrow \frac{du}{dt} = 4e^{4t} + 4e^{4t}\,[\cos^2 3t + \sin^2 3t] = 4e^{4t} + 4e^{4t}\,(1)$$

$$\therefore \frac{du}{dt} = 8\,e^{4t}.$$

Q37. If u = xy + yz + zx where $x = \frac{1}{t}$, $y = e^t$ and $z = e^{-t}$ find $\frac{du}{dt}$.

Answer :

Given that,

$u = xy + yz + zx$ ... (1)

$x = \frac{1}{t}$ ... (2)

$y = e^t$ ... (3)

$z = e^{-t}$ ... (4)

Differentiating equation (1) partially with respect to 'x', 'y', and 'z' respectively

$$\frac{\partial u}{\partial x} = y + z$$

$$\frac{\partial u}{\partial y} = x + z$$

$$\frac{\partial u}{\partial z} = y + x$$

Differentiating equations (2), (3) and (4), partially with respect to 't',

$$\frac{\partial x}{\partial t} = \frac{-1}{t^2} \; ; \; \frac{\partial y}{\partial t} = e^t$$

$$\frac{\partial z}{\partial t} = -\,e^{-t}$$

From the definition of total derivatives,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} \quad ...(5)$$

Substituting the corresponding values in equation (5),

$$\frac{du}{dt} = (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

$$= \frac{-(y+z)}{t^2} + e^t(x+z) - \frac{(y+x)}{e^t}$$

$$\Rightarrow \frac{du}{dt} = \frac{-(y+z)}{t^2} + e^t(x+z) - \frac{(y+x)}{e^t} \quad ...(6)$$

Substituting equations (2), (3) and (4) in equation (6),

$$\frac{du}{dt} = \frac{-(e^t+e^{-t})}{t^2} + e^t\left(\frac{1}{t}+e^{-t}\right) - \frac{\left(e^t+\frac{1}{t}\right)}{e^t}$$

$$= \frac{-2\cosh t}{t^2} + \frac{e^t}{t} + 1 - 1 - \frac{e^{-t}}{t}$$

$$= \frac{-2\cosh t}{t^2} + \frac{2\sinh t}{t}$$

$$\therefore \frac{du}{dt} = \frac{-2\cosh t}{t^2} + \frac{2\sinh t}{t}.$$

3.4 DERIVATIVES OF COMPOSITE AND IMPLICIT FUNCTIONS (CHAIN RULE), CHANGE OF VARIABLES

Q38. Write about,

(i) Derivatives of composite function

(ii) Derivatives of implicit function.

Answer :

(i) Derivatives of Composite Function

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \Psi(t)$, then 'u' is called a composite function of two variables and is expressed as,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

(ii) Derivatives of Implicit Function

A function of the form $f(x, y) = C$ is known as implicit function.

Where,

C - Constant.

The expression for the first differential coefficient of an implicit function is,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Q39. If $u = \sin^{-1}(x - y)$, $x = 3t$ and $y = 4t^3$, then show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Answer :

Given that,

$$u = \sin^{-1}(x-y) \quad ...(1)$$

$$x = 3t \quad ...(2)$$

$$y = 4t^3 \quad ...(3)$$

The above equations represent a composite function of t and is expressed as,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} \quad ...(4)$$

Partially differentiating equation (1), with respect to 'x',

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \quad ...(5)$$

Partially differentiating equation (1) with respect to 'y',

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(-1) \quad ...(6)$$

From equations (2) and (3),

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = 12t^2 \quad ...(7)$$

Substituting equations (5), (6) and (7) in equation (4),

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}.3 + \frac{1}{\sqrt{1-(x-y)^2}}(-1)(12t^2)$$

$$= \frac{3}{\sqrt{1-(x-y)^2}} + \frac{(-12t^2)}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$\therefore \frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}$$

Substituting the values of 'x' and 'y' in above equation,

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(9t^2+16t^6-24t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)}(1-4t^2)}$$

$$= \frac{3}{\sqrt{(1-t^2)}}$$

$$\therefore \frac{du}{dt} = \frac{3}{\sqrt{(1-t^2)}}.$$

Q40. Using implicit differentiation, obtain $\frac{dy}{dx}$ when $f(x, y) = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right) = 0$.

Answer :

Given that,

$$f(x, y) = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right) \quad \text{... (1)}$$

The implicit function is expressed as,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{... (2)}$$

Partially differentiating equation (1) with respect to x and y,

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} + \frac{-\left(\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} \quad ; \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} \quad ; \quad \frac{\partial f}{\partial y} = \frac{2y+x}{x^2+y^2}$$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{2x-y}{x^2+y^2}$$

Substituting the corresponding values in equation (2),

$$\frac{dy}{dx} = \frac{-\frac{2x-y}{x^2+y^2}}{\frac{2y+x}{x^2+y^2}} = \frac{y-2x}{2y+x}$$

$$\therefore \frac{dy}{dx} = \frac{y-2x}{2y+x}$$

Q41. Given the transformations u = e^xcosy and v = e^xsiny and that ϕ is a function of u and v and also of x and y. Prove that $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right)$.

Answer :

Given transformations are,

$u = e^x \cos y$; $v = e^x \sin y$

Differentiating above equations with respect to x and y,

$$\frac{\partial u}{\partial x} = e^x \cos y \;;\; \frac{\partial u}{\partial y} = -\, e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \;;\; \frac{\partial v}{\partial y} = e^x \cos y$$

Consider, $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial \phi}{\partial v}\dfrac{\partial v}{\partial x}$ [$\because$ From chain rule]

$\Rightarrow \quad \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial u} e^x \cos y + \dfrac{\partial \phi}{\partial v} e^x \sin y$

$\Rightarrow \quad \dfrac{\partial \phi}{\partial x} = u\dfrac{\partial \phi}{\partial u} + v\dfrac{\partial \phi}{\partial v}$ ... (1)

$\dfrac{\partial}{\partial x} = u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v}$ ... (2)

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial x}\right]$$

$= \left(u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v}\right)\left(u\dfrac{\partial \phi}{\partial u} + v\dfrac{\partial \phi}{\partial v}\right)$ [$\because$ From equations (1) and (2)]

$\therefore \quad \dfrac{\partial^2 \phi}{\partial x^2} = u^2\dfrac{\partial^2 \phi}{\partial u^2} + uv\dfrac{\partial^2 \phi}{\partial u\,\partial v} + uv\dfrac{\partial^2 \phi}{\partial u\,\partial v} + v^2\dfrac{\partial^2 \phi}{\partial v^2}$ ... (3)

Consider, $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \phi}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial \phi}{\partial v}\dfrac{\partial v}{\partial y}$

$= \dfrac{\partial \phi}{\partial u}\left(-\, e^x \sin y\right) + \dfrac{\partial \phi}{\partial v}\left(e^x \cos y\right)$

$\Rightarrow \quad \dfrac{\partial \phi}{\partial y} = -\, v\dfrac{\partial \phi}{\partial u} + u\dfrac{\partial \phi}{\partial v}$ ... (4)

$\Rightarrow \quad \dfrac{\partial}{\partial y} = -\, v\dfrac{\partial}{\partial u} + u\dfrac{\partial}{\partial v}$ ... (5)

Differentiating equation (4) partially with respect to 'y',

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$$

$= \left(-\, v\dfrac{\partial}{\partial u} + u\dfrac{\partial}{\partial v}\right)\left(-\, v\dfrac{\partial \phi}{\partial u} + u\dfrac{\partial \phi}{\partial v}\right)$ [$\because$ From equations 4 and 5]

$\therefore \quad \dfrac{\partial^2 \phi}{\partial y^2} = v^2\dfrac{\partial^2 \phi}{\partial u^2} - uv\dfrac{\partial \phi}{\partial u\,\partial v} - uv\dfrac{\partial \phi}{\partial u\,\partial v} - u^2\dfrac{\partial^2 \phi}{\partial v^2}$ ... (6)

Adding equations (3) and (6),

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = u^2\frac{\partial^2 \phi}{\partial u^2} + uv\frac{\partial^2 \phi}{\partial u\,\partial v} + uv\frac{\partial^2 \phi}{\partial u\,\partial v} + v^2\frac{\partial^2 \phi}{\partial v^2} + v^2\frac{\partial^2 \phi}{\partial u^2} - uv\frac{\partial \phi}{\partial u\,\partial v} - uv\frac{\partial \phi}{\partial u\,\partial v} + u^2\frac{\partial^2 \phi}{\partial v\ 2}$$

$$= \frac{\partial^2 \phi}{\partial u^2}(u^2 + v^2) + \frac{\partial^2 \phi}{\partial v^2}(v^2 + u^2)$$

$$= (u^2 + v^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right).$$

Q42. If u = log ($x^2 + y^3 + z^3 - 3xyz$). Show that $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$.

Answer :

Given function is,

$u = \log(x^2 + y^3 + z^3 - 3xyz)$ ... (1)

Differentiating equation (1) partially with respect to 'x', 'y' and 'z'

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3x^2 - 3yz) \quad \text{... (2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3y^2 - 3xz) \quad \text{... (3)}$$

$$\frac{\partial u}{\partial z} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3z^2 - 3xy) \quad \text{... (4)}$$

Adding equations (2), (3) and (4),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3y^2 + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x + y + z} \quad \text{... (5)}$$

Differentiating equation (5) partially with respect to 'x', 'y' and 'z'

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} \quad \text{... (6)}$$

$$\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} \quad \text{... (7)}$$

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} \quad \text{... (8)}$$

Adding equations (6), (7) and (8),

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} + \left(\frac{-3}{(x + y + z)^2}\right) + \left(\frac{-3}{(x + y + z)^2}\right)$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}$$

$$\therefore \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}.$$

Q43. Explain change of variables in partial differentiation.

Answer :

Let u be a function of x and y given as,

$u = f(x, y)$

Where, x, y are functions of s, t

$x = \phi(s, t)$

$y = \psi(s, t)$

If t is considered as a constant then, x, y, u are the functions of 's' only.

From chain rule,

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

The ordinary derivatives are replaced by partial derivatives because x, y are functions of variables s and t.

$$\Rightarrow \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \quad \text{... (1)}$$

If s is considered as a constant then x, y, u will be functions of t only

$$\Rightarrow \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad \text{... (2)}$$

Equations (1) and (2) represent simultaneous equations in $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

The values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are obtained in terms of $\frac{\partial^2}{\partial s}, \frac{\partial u}{\partial t}$, u, s, t by solving equations (1) and (2)

Hence, the variables x, y are changed to s, t.

If the variables s and t are expressed in terms of x and y i.e.,

$s = \xi(x, y)$ and $t = \eta(x, y)$ then the following formulae are obtained

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

The above two equations are solved to obtain the values of $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$

Hence, the variables s, t are changed to x, y.

Q44. If Z = f(x, y), x = $e^{2u} + e^{-2v}$, y = $e^{-2u} + e^{2v}$, then show that, $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$.

Answer :

Given that,

$$z = f(x, y)$$

$$x = e^{2u} + e^{-2v}, \; y = e^{-2u} + e^{2v}$$

Differentiating above equations partially with respect to u and v,

$$\frac{\partial x}{\partial u} = 2e^{2u} \qquad \frac{\partial y}{\partial u} = -2e^{-2u}$$

$$\frac{\partial x}{\partial v} = -2e^{-2v} \qquad \frac{\partial y}{\partial v} = 2e^{2v}$$

From Chain rule,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u}$$

$$\therefore \quad \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}(2e^{2u}) + \frac{\partial f}{\partial y}(-2e^{-2u}) \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}(-2e^{-2v}) + \frac{\partial f}{\partial y}(2e^{2v}) \qquad \text{... (2)}$$

Subtracting equation (2) from equation (1),

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}(2e^{2u} + 2e^{-2v}) + \frac{\partial f}{\partial y}(-2e^{-2u} - 2e^{2v})$$

$$\Rightarrow \quad \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}$$

$$\therefore \quad \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Q45. If x = u + v + w, y = uv + vw + wu, z = uvw, and f is a function of x, y, z then show that,

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}.$$

Answer :

Given that,

f is a function of x, y, z

$x = u + v + w \qquad y = uv + vw + wu \qquad z = uvw$

Differentiating above equations partially with respect to u, v and w,

$$\frac{\partial x}{\partial u}=1 \quad ; \quad \frac{\partial y}{\partial u}=v+w \quad ; \quad \frac{\partial z}{\partial u}=vw$$

$$\frac{\partial x}{\partial v}=1 \quad ; \quad \frac{\partial y}{\partial v}=u+w \quad ; \quad \frac{\partial z}{\partial v}=uw$$

$$\frac{\partial x}{\partial w}=1 \quad ; \quad \frac{\partial y}{\partial w}=u+v \quad ; \quad \frac{\partial z}{\partial w}=uv$$

From chain rule,

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial u}$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}(1)+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}(vw)$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial v}$$

$$\Rightarrow \quad \frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}(1)+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}(uw)$$

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}.\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial w}$$

$$\Rightarrow \quad \frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}(1)+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}(uv)$$

Consider $u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}$

$$= u\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}(vw)\right) + v\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}(uw)\right) + w\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}(uv)\right)$$

$$= u\frac{\partial f}{\partial x}+u(v+w)\frac{\partial f}{\partial y}+uvw\frac{\partial f}{\partial z}+v\frac{\partial f}{\partial x}+v(u+w)\frac{\partial f}{\partial y}+uvw\frac{\partial f}{\partial z}+w\frac{\partial f}{\partial x}+w(u+v)\frac{\partial f}{\partial y}+uvw\frac{\partial f}{\partial z}$$

$$= (u+v+w)\frac{\partial f}{\partial x}+2(uv+vw+wu)\frac{\partial f}{\partial y}+3uvw\frac{\partial f}{\partial z}$$

$$= x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}$$

$$\therefore \quad u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}.$$

Q46. If z = f(x, y), where x = e^u cos v and y = e^u sin v, then show that $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$

Answer :

Given functions are,

$$z = f(x, y)$$

$$x = e^u \cos v \quad \text{... (1)}$$

$$y = e^u \sin v \quad \text{... (2)}$$

Differentiating equation (1) partially with respect to u and v,

$$\frac{\partial x}{\partial u} = e^u \cos v, \ \frac{\partial x}{\partial v} = -e^u \sin v$$

Differentiating equation (2) partially with respect to u and v,

$$\frac{\partial y}{\partial u} = e^u \sin v, \ \frac{\partial y}{\partial v} = e^u \cos v$$

From chain rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} \quad \text{... (3)}$$

Substituting the corresponding values in equation (3),

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)$$

Multiplying on both sides by 'y',

$$y\frac{\partial z}{\partial u} = y\left[\frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)\right]$$

$$= e^u \sin v\left[e^u \cos v\frac{\partial z}{\partial x} + e^u \sin v\frac{\partial z}{\partial y}\right] \quad [\because \text{From equation (2)}]$$

$$\therefore \quad y\frac{\partial z}{\partial u} = e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\sin^2 v\frac{\partial z}{\partial y} \quad \text{... (4)}$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} \quad \text{... (5)}$$

Substituting the corresponding values in equation (5),

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v)$$

Multiplying on the sides by 'x',

$$x\frac{\partial z}{\partial v} = x\left[-e^u \sin v\frac{\partial z}{\partial x} + e^u \cos v\frac{\partial z}{\partial y}\right]$$

$$= e^u \cos v\left[-e^u \sin v\frac{\partial z}{\partial x} + e^u \cos v\frac{\partial z}{\partial y}\right] \quad [\because \text{From equation (1)}]$$

$$\therefore \quad x\frac{\partial z}{\partial v} = -e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\cos^2 v\frac{\partial z}{\partial y} \quad \text{... (5)}$$

Adding equations (4) and (5),

$$x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = -e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\cos^2 v\frac{\partial z}{\partial y} + e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\sin^2 v\frac{\partial z}{\partial y}$$

$$= e^{2u}(\sin^2 v + \cos^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y}$$

$$\therefore \ x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$$

3.5 JACOBIAN

Q47. If x + y + z = u, y + z = uv, z = uvw show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

Answer :

Given functions are,

$$u = x + y + z \quad \text{... (1)}$$

$$uv = y + z \quad \text{... (2)}$$

$$uvw = z \quad \text{... (3)}$$

Substituting the value of z from equation (3), in equation (1),

$$uv = y + uvw$$

$$\Rightarrow \quad y = uv - uvw$$

$$\Rightarrow \quad y = uv[1 - w] \quad \text{... (4)}$$

Substituting the values of z and y from equations (3) and (4) in equation (1),

$$u = x + uv[1-w] + uvw$$

$$\Rightarrow \quad u = x + uv - uvw + uvw$$

$$\Rightarrow \quad u = x + uv$$

$$\Rightarrow \quad x = u - uv$$

$$\Rightarrow \quad x = u(1 - v) \quad \text{... (5)}$$

Differentiating equation (5) with respect to 'u',

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}[u(1-v)]$$

$$\Rightarrow \quad \frac{\partial x}{\partial u} = \frac{\partial}{\partial u}[u - uv] = 1 - v$$

Differentiating above equation with respect to 'v',

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}[u - uv] = 0 - u$$

$$\Rightarrow \quad \frac{\partial x}{\partial v} = -u$$

Differentiating above equation with respect to 'w',

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}[u - uv]$$

$$\frac{\partial x}{\partial w} = 0$$

Differentiating equation 4 with respect to u, v and w,

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}[uv(1 - w)] = \frac{\partial}{\partial u}[uv - uvw] = v - vw$$

$$\Rightarrow \quad \frac{\partial y}{\partial u} = v[1 - w]$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}[uv - uvw]$$

$$= u - uw$$

$$\Rightarrow \quad \frac{\partial y}{\partial v} = u[1 - w]$$

$$\frac{\partial y}{\partial w} = \frac{\partial}{\partial w}[uv - uvw]$$

$$= 0 - uv$$

$$\Rightarrow \quad \frac{\partial y}{\partial w} = -uv$$

Differentiating equation (3) with respect to u, v and w,

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}[uvw]$$

$$\Rightarrow \quad \frac{\partial z}{\partial u} = vw$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}[uvw]$$

$$\Rightarrow \quad \frac{\partial z}{\partial v} = uw$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}[uvw]$$

$$\Rightarrow \quad \frac{\partial z}{\partial w} = uv$$

The Jacobian of x, y, z is given as,

$$J(x, y, z) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \text{... (6)}$$

Substituting the corresponding values in equation (6),

$$J(x, y, z) = \begin{vmatrix} (1-v) & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)[(u(1-w)(uv) - (uw \times -uv))]$$

$$- (-u)[v(1-w)(uv) - (vw \times -uv)] + 0$$

$$= (1-v)[u(uv - uvw) + u^2vw] + u[v(uv - uvw) + uv^2w]$$

$$= (1-v)[u^2v - u^2vw + u^2vw] + u[uv^2 - uv^2w + uv^2w]$$

$$= (1-v)[u^2v] + u[uv^2]$$

$$= (1-v)(u^2v) + u^2v^2$$

$$= u^2v[1 - v + v]$$

$$= u^2v[1 + 0]$$

$$= u^2v$$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

Q48. Prove that u = $x\sqrt{1-y^2}+y\sqrt{1-x^2}$, v = $\sin^{-1}(x)+\sin^{-1}(y)$ are functionally dependent and find the relation between them.

Answer :

Given functions are,

$$u = x\sqrt{1-y^2}+y\sqrt{1-x^2} \quad \text{... (1)}$$

$$v = \sin^{-1}(x)+\sin^{-1}(y) \quad \text{... (2)}$$

u and *v* are said to be functionally dependent if they satisfy the following condition,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

Partially differentiating equation (1) with respect to 'x',

$$\frac{\partial u}{\partial x} = \sqrt{1-y^2} - 2xy\frac{1}{2\sqrt{1-x^2}}$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

Partially differentiating equation (1) with respect to 'y',

$$\frac{\partial u}{\partial y} = -2yx\frac{1}{2\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

Partially differentiating equation (2) with respect to 'x',

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-x^2}} + 0$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Partially differentiating equation (2) with respect to 'y',

$$\frac{\partial v}{\partial y} = 0 + \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \sqrt{1-y^2}-\frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}}+\sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= \left[\sqrt{1-y^2}-\frac{xy}{\sqrt{1-x^2}}\right]\left[\frac{1}{\sqrt{1-y^2}}\right] - \left[\frac{-xy}{\sqrt{1-y^2}}+\sqrt{1-x^2}\right]\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \left[1-\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right] - \left[\frac{-xy}{\sqrt{1-x^2}\sqrt{1-y^2}}+1\right]$$

$$= 1-\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}+\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}-1$$

$$= 0$$

$\therefore$ *u*, *v* are functionally dependent

$$v = \sin^{-1}x + \sin^{-1}y$$

$$= \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

$$\left[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right]$$

$$= \sin^{-1}(u)$$

$\Rightarrow$ $\sin v = u$

$\therefore$ Relation between *u* and *v* is $\sin v = u$.

Q49. Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 if $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$.

OR

If $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$ prove that $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = 4$.

Answer :

Given that,

$$y_1 = \frac{x_2x_3}{x_1} \quad \text{... (1)}$$

$$y_2 = \frac{x_3x_1}{x_2} \quad \text{... (2)}$$

$$y_3 = \frac{x_1x_2}{x_3} \quad \text{... (3)}$$

Partially differentiating equations (1), (2) and (3) with respect to 'x_1'

$$\frac{\partial y_1}{\partial x_1} = x_2x_3\left(\frac{-1}{x_1^2}\right) = -\frac{x_2x_3}{x_1^2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$$

Partially differentiating equations (1), (2) and (3) with respect to 'x_2'

$$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$$

$$\frac{\partial y_2}{\partial x_2} = x_3x_1\left(-\frac{1}{x_2^2}\right) = -\frac{x_1x_3}{x_2^2}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$$

Partially differentiating equation (1), (2) and (3) with respect to 'x_3'

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_3}{\partial x_3} = x_1 x_2\left(\frac{-1}{x_3^2}\right) = \frac{-x_1 x_2}{x_3^2}$$

$$\therefore \quad \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$\Rightarrow \quad \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\left(\frac{-x_1 x_3}{x_2^2}\right)\left(\frac{-x_1 x_2}{x_3^2}\right) - \left(\frac{x_1}{x_2}\right)\left(\frac{x_1}{x_3}\right)\right] - \frac{x_3}{x_1}\left[\frac{x_3}{x_2}\left(\frac{-x_1 x_2}{x_3^2}\right) - \left(\frac{x_1}{x_2}\right)\left(\frac{x_2}{x_3}\right)\right] + \frac{x_2}{x_1}\left[\left(\frac{x_3}{x_2}\right)\left(\frac{x_1}{x_3}\right) - \left(\frac{-x_1 x_3}{x_2^2}\right)\left(\frac{x_2}{x_3}\right)\right]$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\frac{x_1^2}{x_2 x_3} - \frac{x_1^2}{x_2 x_3}\right] - \frac{x_3}{x_1}\left[\frac{-x_1}{x_3} - \frac{x_1}{x_3}\right] + \frac{x_2}{x_1}\left[\frac{x_1}{x_2} + \frac{x_1}{x_2}\right]$$

$$= -\frac{x_2 x_3}{x_1^2}[0] - \frac{x_3}{x_1}\left[-\frac{2x_1}{x_3}\right] + \frac{x_2}{x_1}\left[\frac{2x_1}{x_2}\right]$$

$$= 2 + 2 = 4$$

$$\therefore \quad \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4.$$

Q50. If x = r sinθ cos φ, y = r sinθ sin φ, z = rcosθ, find $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

Answer :

Given that,

$x = r\sin\theta \cos\phi$ ... (1)

$y = r \sin\theta \sin\phi$ ... (2)

$z = r\cos\theta$ ... (3)

Partially differentiating equations (1), (2) and (3) with respect to 'r'.

$$\frac{\partial x}{\partial r} = \sin\theta \cos\phi \ ; \ \frac{\partial y}{\partial r} = \sin\theta \sin\phi \ ; \ \frac{\partial z}{\partial r} = \cos\theta$$

Partially differentiating equations (1), (2) and (3) with respect to 'θ'.

$$\frac{\partial x}{\partial \theta} = r\cos\theta \cos\phi \ ; \ \frac{\partial y}{\partial \theta} = r\cos\theta \sin\phi \ ; \ \frac{\partial z}{\partial \theta} = -r\sin\theta$$

Partially differentiating equations (1), (2) and (3) with respect to 'φ'.

$$\frac{\partial x}{\partial \phi} = -r\sin\theta \sin\phi \ ; \ \frac{\partial y}{\partial \phi} = r\sin\theta \cos\phi \ ; \ \frac{\partial y}{\partial \phi} = 0$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi\,[r\cos\theta\sin\phi(0) + r\sin\theta(r\sin\theta\cos\phi)] - r\cos\theta\cos\phi\,[\sin\theta\sin\phi(0) - \cos\theta(r\sin\theta\cos\phi)]$

$\quad - r\sin\theta\sin\phi\,[\sin\theta\sin\phi\,(-r\sin\theta) - \cos\theta(r\cos\theta\sin\phi)]$

$= \sin\theta\cos\phi\,[r\sin\theta(r\sin\theta\cos\phi) - [r\cos\theta\cos\phi]$

$\quad [-\cos\theta(r\sin\theta\cos\phi)] - r\sin\theta\sin\phi\,[-r\sin\theta(\sin\theta\sin\phi) - \cos\theta(r\cos\theta\sin\phi)]$

$= \sin\theta\cos\phi[r^2\sin^2\theta\cos\phi] - r\cos\theta\cos\phi[-r\sin\theta\cos\theta\cos\phi] - r\sin\theta\sin\phi\,[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$

$= r^2\sin^3\theta\,[\cos^2\phi + \sin^2\phi] + r^2\sin\theta\cos^2\theta\,[\cos^2\phi + \sin^2\phi]$

$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta \qquad [\because \sin^2\phi + \cos^2\phi = 1]$

$= r^2\sin\theta\,[\sin^2\theta + \cos^2\theta]$

$= r^2\sin\theta$

$$\therefore \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta.$$

3.6 HIGHER ORDER PARTIAL DERIVATIVES

Q51. Explain the concept of finding the higher order derivatives of second and third order with examples.

Answer :

Higher Order Derivatives

The derivatives of the principle derivative $\frac{dy}{dx}$ are known as higher order derivatives.

(i) Second Order Derivative

The derivative of first order derivative $\left(\text{i.e.,} \frac{dy}{dx}\right)$ is known as second derivative or second order derivative.

$$\text{i.e., } \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dx^2}$$

The different notations of second order derivative are,

1. D^2y
2. y^2
3. y''
4. f''

Example

If y = x^3, then $\frac{d^2y}{dx^2}$

Given that,

$y = x^3$ `...(1)

Differentiating equation (1) with respect to 'x',

$$\frac{dy}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(x^3) = 3x^{3-1} \qquad \left[\because \frac{d}{dx}x^n = nx^{n-1}\right]$$

$= 3x^2$ ...(2)

Differentiating equation (2) with respect to 'x',

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}3x^2$$

$$= 3\frac{d}{dx}x^2 = 3\,(2x^{2-1})$$

$$\frac{d^2y}{dx^2} = 6x$$

$$\therefore \frac{d^2y}{dx^2} = 6x$$

(ii) Third Order Derivative

The derivative of second order derivative $\left(\text{i.e.,} \frac{d^2y}{dx^2}\right)$ is known as third order derivative.

i.e., $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

The different notations of third order derivatives are,

1. D^3y
2. y_3
3. y'''
4. f'''

Example

If $y = x^4$, then $\frac{d^3y}{dx^3}$

Given that,

$$y = x^4 \quad \text{... (1)}$$

Differentiating equation (1) with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}\; x^4$$

$$= 4x^{4-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = 4x^3 \quad \text{... (2)}$$

Differentiating equation (2) with respect to 'x',

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3)$$

$$= 4 \times 3x^{3-1}$$

$$\frac{d^2y}{dx^2} = 12x^2 \quad \text{... (3)}$$

Differentiating equation (3) with respect to 'x',

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(12x^2)$$

$$\frac{d^3y}{dx^3} = 12\,(2x^{2-1})$$

$$= 24x$$

$$\therefore \frac{d^3y}{dx^3} = 24x$$

Q52. If y = ae^x + be^{-x}, show that $\frac{d^2y}{dx^2} - y = 0$.

Answer :

Given equation is,

$$y = ae^x + be^{-x} \quad \text{... (1)}$$

Differentiating equation (1) on both sides with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}[ae^x + be^{-x}]$$

$\Rightarrow \quad \frac{dy}{dx} = a\ [e^x] + b\ (e^{-x}(-1))$

$\Rightarrow \quad \frac{dy}{dx} = ae^x - be^{-x}$ ... (2)

Again differentiating equation (2) with respect to x,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[ae^x - be^{-x}] \qquad \begin{bmatrix} \frac{d}{dx}(e^x) = e^x \\ \frac{d}{dx}(e^{-x}) = e^{-x}(-1) \end{bmatrix}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = a[e^x] - b\ [e^{-x}(-1)]$

$\Rightarrow \quad \frac{d^2y}{dx^2} = ae^x + be^{-x}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = y$ [$\because$ From equation (1)]

$\therefore \frac{d^2y}{dx^2} - y = 0$

Q53. If y = a cos (logx) + b sin (logx), show that $x^2y_2 + xy_1 + y = 0$

Answer :

Given function is,

$y = a\cos(\log x) + b\sin(\log x)$ ... (1)

Differentiating equation (1) on both sides with respect to x,

$\frac{dy}{dx} = a\frac{d}{dx}[\cos(\log x)] + b\frac{d}{dx}[\sin(\log x)]$

$\Rightarrow \quad \frac{dy}{dx} = a\frac{d}{dx}[\cos(\log x)]\frac{d}{dx}(\log x) + b\cos(\log x)\frac{d}{dx}(\log x)$

$\Rightarrow \quad \frac{dy}{dx} = -a\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$

$\Rightarrow \quad x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$ ...(2)

Differentiating equation (2) on both sides with respect to x,

$x\frac{d}{dx}\left[\frac{dy}{dx}\right] + \frac{dy}{dx} \times \frac{d}{dx}(x) = -a\cos(\log x) \times \frac{1}{x} - b\sin(\log x) \times \frac{1}{x}$

$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times (1) = -\left[\frac{a\cos(\log x) + b\sin(\log x)}{x}\right]$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$ [$\because$ From equation(1)]

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ ...(3)

Since, $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

Thus, equation (3) reduces to,

$x^2y_2 + xy_1 + y = 0$

3.7 TAYLOR'S SERIES OF FUNCTIONS OF TWO VARIABLES

Q54. Prove Taylor's series for the functions of two variables.

Answer :

Let, $f(x + h, y + k)$ be a function of one variable i.e., x.

Then according to the Taylor's theorem of a single variable,

$$f(x+h, y+k) = f(x, y+k) + \frac{h}{1!}\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \ldots \quad \ldots (1)$$

If $f(x, y + k)$ is a function of y, then,

$$f(x, y+k) = f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots + \frac{h}{1!}\frac{\partial}{\partial x}\left[f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right] \quad \ldots (2)$$

Substituting equation (2) in equation (1),

$$f(x+h, y+k) = f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right]$$

$$= f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \frac{h}{1!}\frac{\partial f(x, y)}{\partial x} + hk\frac{\partial^2 f(x, y)}{\partial x \partial y} + \frac{hk^2}{2!}\frac{\partial^3 f(x, y)}{\partial x \partial y^2} + \frac{h^2}{2!}\frac{\partial^2 f(x, y)}{\partial x^2}$$

$$+ \frac{h^2 k}{2!}\frac{\partial^3 f(x, y)}{\partial x^2 \partial y} + \frac{h^2 k^2}{2!2!}\frac{\partial^4 f(x, y)}{\partial x^2 \partial y^2} + \ldots$$

Neglecting higher order derivatives,

$$f(x+h, y+k) = f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \frac{h}{1!}\frac{\partial f(x, y)}{\partial x} + hk\frac{\partial^2 f(x, y)}{\partial x \partial y} + \frac{h^2}{2!}\frac{\partial^2 f(x, y)}{\partial x^2} + \ldots$$

$$= f(x, y) + \frac{h}{1!}\frac{\partial f(x, y)}{\partial x} + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{1}{2!}\left[h^2\frac{\partial^2 f(x, y)}{\partial x^2} + k^2\frac{\partial^2 f(x, y)}{\partial y^2} + 2hk\frac{\partial^2 f(x, y)}{\partial x \partial y}\right] + \ldots$$

$$= f(x, y) + \left[\frac{h}{1!}f_x(x, y) + \frac{k}{1!}f_y(x, y)\right] + \frac{1}{2!}\left[h^2 f_{xx}(x, y) + k^2 f_{yy}(x, y) + 2hkf_{xy}(x, y)\right] + \ldots$$

Let,

$x = a$ and $y = b$, then,

$$f(a+h, b+k) = f(a, b) + \left[\frac{h}{1!}f_x(a,b) + \frac{k}{1!}f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hkf_{xy}(a,b)\right] + \ldots$$

Substituting $a + h = x$ and $b + k = y$ such that $h = x - a$ and $k = y - b$ in the above equation,

$$f(x, y) = f(a, b) + \left[\frac{(x-a)}{1!}f_x(a,b) + \frac{(y-b)}{1!}f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a, b) + (y-b)^2 f_{yy}(a, b) + 2(x-a)(y-b)f_{xy}(a, b)\right] + \ldots$$

The above equation is the Taylor's theorem which is used to expand $f(x, y)$ in (a, b) i.e., two variables.

Q55. Obtain the Taylor series expansion of the function $f(x, y) = e^{2x+y}$ about (0, 0) upto third degree terms.

Answer : June/July-17, Q14(b)

Given function is,

$$f(x, y) = e^{2x+y} \quad \ldots (1)$$

$$f(0,0) = e^{2(0)+0} = 1$$

Differentiating equation (1) partially with respect to x and y.

$f_y(x, y) = e^{2x+y}$ $\quad f_y(0,0) = e^{0+0} = 1$

$f_{yy}(x, y) = e^{2x+y}$ $\quad f_{yy}(0,0) = e^{0+0} = 1$

$f_x(x, y) = 2e^{2x+y}$ $\quad f_x(0,0) = 2e^{0+0} = 2$

$f_{xx}(x, y) = 4e^{2x+y}$ $\quad f_{xx}(0,0) = 4e^{0+0} = 4$

$f_{yx}(x, y) = 2e^{2x+y}$ $\quad f_{yx}(0,0) = 2e^{0+0} = 2$

$f_{xy}(x, y) = 2e^{2x+y}$ $\quad f_{xy}(0,0) = 2e^{0+0} = 2$

$f_{xyy}(x, y) = 2e^{2x+y}$ $\quad f_{xyy}(0,0) = 2e^{0+0} = 2$

$f_{xxy}(x, y) = 4e^{2x+y}$ $\quad fx_{xy}(0,0) = 4e^{0+0} = 2$

$f_{xxx}(x, y) = 8e^{2x+y}$ $\quad f_{xxx}(0,0) = 8e^{0+0} = 8$

$f_{yyy}(x, y) = e^{2x+y}$ $\quad f_{yyy}(0,0) = e^{0+0} = 1$

Taylor's series is given as,

$$f(x, y) = f(x_0, y_0) + (x - x_0)\, f_x(x_0, y_0) + (y - y_0) f_y\,(x_0, y_0) + \frac{1}{2!}\,[(x - x_0)^2\, f_{xx}\,(x_0, y_0) + 2(x - x_0)\,(y - y_0)\, f_{xy}\,(x_0, y_0) +$$

$$(y - y_0)^2 f_{yy}\,(x_0, y_0) + \frac{1}{3!}\,[(x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2\,(y - y_0) f_{xxy}\,(x_0, y_0)$$

$$+ 3(x - x_0)\,(y - y_0)^2 f_{xyy}\,(x_0, y_0) + (y - y_0)^3 f_{yyy}\,(x_0, y_0)]$$

$$= f(0, 0) + (x - 0) f_x(0, 0) + (y - 0) f_y\,(0, 0) + \frac{1}{2!}\,[(x - 0)^2\, f_{xx}\,(0, 0) +$$

$$2(x - 0)\,(y - 0) f_{xy}\,(0, 0) + (y - 0)^2 f_{yy}\,(0, 0)] + \frac{1}{3!}\,[(x - 0)^3 f_{xxx}(0, 0)$$

$$+ 3(x - 0)^2 (y - 0) f_{xxy}\,(0, 0) + 3\,(x - 0)\,(y - 0)^2 f_{xyy}\,(0, 0) + (y - 0)^3 f_{yyy}(0, 0)] \quad \text{... (2)}$$

Substituting the corresponding values in equation (2),

$$= 1 + x(2) + y(1) + \frac{1}{2}\,[x^2(4) + 2xy\,(2) + y^2\,(1)] + \frac{1}{6}\,[x^3(8) + 3x^2y(4) + 3xy^2\,(2) + y^3(1)]$$

$$= 1 + 2x + y + \frac{1}{2}\,[4x^2 + 4xy + y^2] + \frac{1}{6}\,[8x^3 + 12x^2y + 6xy^2 + y^3]$$

$$= 1 + 2x + y + 2x^2 + 2xy + \frac{y^2}{2} + \frac{4}{3}\,x^3 + 2x^2y + xy^2 + \frac{y^3}{6}$$

$$\therefore \quad e^{2x+y} = 1 + 2x + y + 2x^2 + 2xy + \frac{y^2}{2} + \frac{4}{3}\,x^3 + 2x^2y + xy^2 + \frac{y^3}{6}.$$

Q56. Obtain the Taylor's series expansion of $x^3 + y^3 + xy^2$ in terms of powers of (x – 1) and (y – 2) up to third degree terms.

Answer :

Given function is,

$$f(x, y) = x^3 + y^3 + xy^2$$

Taylor series is given by,

$$f(x, y) = f(a, b) + \frac{1}{1!}\,[(x - a) f_x(a, b) + (y - b) f_y\,(a - b)]$$

$$+ \frac{1}{2!}\,[(x - a)^2 f_{xx}\,(a, b) + 2\,(x - a)\,(y - b) f_{xy}\,(a, b)$$

$$+ (y - b)^2 f_{yy}\,(a, b)] + \frac{1}{3!}\,[(x - a)^3 f_{xxx}\,(a, b) + 3(x - a)^2\,(y - b) f_{xyy}\,(a - b) + 3(x - a)\,(y - b)^2 f_{xyy}\,(a, b)$$

$$+ (y - b)^3\; f_{yyy}\,(a - b)] \quad \text{... (1)}$$

Here, $a = 1, b = 2$

$f(x, y) = x^3 + y^3 + xy^2$

$\Rightarrow$ $f(1, 2) = (1)^3 + (2)^3 + (1)\ (2)^2 = 13$

$f_x(x, y) = 3x^2 + y^2$

$\Rightarrow$ $f_x(1, 2) = 3(1)^2 + (2)^2 = 7$

$f_y(x, y) = 3y^2 + 2xy$

$\Rightarrow$ $f_y(1, 2) = 3(2)^2 + 2(1)\ (2) = 16$

$f_{xy}(x, y) = 2y$

$\Rightarrow$ $f_{xy}\ (1, 2) = 2(2) = 4$

$f_{xx}\ (x, y) = 6x$

$\Rightarrow$ $f_{xx}\ (1, 2) = 6(1) = 6$

$f_{xxx}\ (x, y) = 6$

$\Rightarrow$ $f_{xxx}(1.\ 2) = 6$

$f_{xxy}\ (x, y) = 0$

$\Rightarrow$ $f_{xxy}\ (1, 2) = 0$

$f_{xyy}\ (x, y) = 2$

$\Rightarrow$ $f_{xyy}\ (1, 2) = 2$

$f_{yy}\ (x, y) = 6y + 2x$

$\Rightarrow$ $f_{yy}\ (1, 2) = 6(2) + 2(1) = 14$

$f_{yyy}\ (x, y) = 6$

$\Rightarrow$ $f_{yyy}\ (1, 2) = 6$

Substituting the corresponding values in equaiton (1),

$$x^3 + y^3 + xy^2 = 13 + \frac{1}{1!}[(x-1)7 + (y-2)16] + \frac{1}{2!}[(x-1)^2\,6 + 2(x-1)\,(x-2)\,(4) + (y-2)^2\,14]$$

$$+ \frac{1}{3!}[(x-1)^3\,6 + 3(x-1)^2\,(y-2)\,(0) + 3(x-1)\,(y-2)^2\,(2) + (y-2)^3\,(6)]$$

$$= 13 + 7(x-1) + 16\,(y-2) + \frac{1}{2}[6\,(x-1)^2 + 8\,(x-1)\,(y-2) + 14\,(y-2)^2] + \frac{1}{6}[6(x-1)^3$$

$$+ 0 + 6(x-1)\,(y-2)^2 + 6\,(y-2)^3]$$

$$= 13 + 7(x-1) + 16\,(y-2) + 3(x-1)^2 + 4\,(x-1)\,(y-2) + 7(y-2)^2 + (x-1)^3 + (x-1)\,(y-2)^2 + (y-2)^3$$

$$\therefore\ x^3 + y^3 + xy^2 = 13 + 7\,(x-1) + 16\,(y-2) + 3(x-1)^2 + 4(x-1)\,(y-2) + 7\,(y-2)^2 + (x-1)^3 + (x-1)\,(y-2)^2 + (y-2)^3$$

Q57. Using Taylor's series expansion, expand e^x siny in powers of x and y as far as terms of the 3rd degree.

OR

Expand e^x sin (y) in powers of x and y upto the third degree terms.

OR

Expand e^xsiny in powers of x and y as far as the terms of the 3rd degree using Taylor's expansion.

Answer :

Let the given function be,

$f(x, y) = e^x \sin y$

$\Rightarrow f(0, 0) = 0$

Consider,

$f_x(x, y) = e^x \sin y \quad ; \quad f_x(0, 0) = 0$

$f_{xy}(x, y) = e^x \cos y \quad ; \quad f_{xy}(0, 0) = 1$

$f_{xx}(x, y) = e^x \sin y \quad ; \quad f_{xx}(0, 0) = 0$

$f_{xyy}(x, y) = -e^x \sin y \, ; \quad f_{xyy}(0, 0) = 0$

$f_{xxx}(x, y) = e^x \sin y \quad ; \quad f_{xxx}(0, 0) = 0$

$f_{xxy}(x, y) = e^x \cos y \quad ; \quad f_{xxy}(0, 0) = 1$

$f_y(x, y) = e^x \cos y \quad ; \quad f_y(0, 0) = 1$

$f_{yy}(x, y) = -e^x \sin y \quad ; \quad f_{yy}(0, 0) = 0$

$f_{yyx}(x, y) = -e^x \sin y \, ; \quad f_{yyx}(0, 0) = 0$

$f_{yyy}(x, y) = -e^x \cos y \quad ; \quad f_{yyy}(x, y) = -1$

From Taylor's series,

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0)$$
$$+ 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + ...$$

$$= 0 + x(0) + y(1) + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)] + \frac{1}{3!}[x^3(0) + 3x^2 y(1) + 3xy^2(0) + y^3(-1)] + ...$$

$$\therefore \quad e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + ...$$

Q58. Expand $\tan^{-1}\left(\frac{y}{x}\right)$ as a Taylor series about the point (1, 1) upto 2nd degree terms.

Answer :

Given function is,

$$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$f(1, 1) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$f_x(x, y) = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{\frac{-y}{x^2}}{\frac{x^2 + y^2}{x^2}}$$

$$\Rightarrow \quad f_x(x, y) = \frac{-y}{x^2 + y^2}$$

$$\Rightarrow \quad f_x(1, 1) = \frac{-1}{1+1} = \frac{-1}{2}$$

$$f_y(x, y) = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{1}{x}\,\frac{1}{\frac{x^2 + y^2}{x^2}} = \frac{x^2}{x(x^2 + y^2)} = \frac{x}{x^2 + y^2}$$

$$\Rightarrow \quad f_y(1, 1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f_{xx}(x, y) = \frac{-y}{(x^2 + y^2)^2}(-2x) = \frac{2xy}{(x^2 + y^2)^2}$$

$$\Rightarrow \quad f_{xx}(1, 1) = \frac{2(1)(1)}{(1+1)^2} = \frac{2}{(2)^2} = \frac{2}{4} = \frac{1}{2}$$

$$f_{xy}(x, y) = \frac{-\left[\left(x^2+y^2\right)1 - y(2y)\right]}{\left(x^2+y^2\right)^2}$$

$$= \frac{-\left[x^2+y^2-2y^2\right]}{\left(x^2+y^2\right)^2}$$

$$f_{xy}(x, y) = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow \quad f_{xy}(1, 1) = \frac{1-1}{(1+1)^2} = 0$$

$$f_{yy}(x, y) = \frac{x}{\left(x^2+y^2\right)^2}(-2y)$$

$$= \frac{-2xy}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow \quad f_{yy}(1, 1) = \frac{-2}{(1+1)^2} = \frac{-2}{4} = -\frac{1}{2}$$

From Taylor's series,

$$f(x, y) = f(1,1) + \frac{1}{1!}\left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + (x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{(x-1)}{2} + \frac{(y-1)}{2} + \frac{1}{2}\left[\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2}\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

Q59. Expand sin (xy) in powers of (x – 1) and $\left(y - \frac{\pi}{2}\right)$ up to second degree terms by using Taylor's series.

OR

Expand sin xy at $\left(1.\frac{\pi}{2}\right)$ upto second degree terms using Taylor's series.

Answer :

Given function is,

$$f(x, y) = \sin(xy)$$

$$\Rightarrow \quad f\left(1, \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$f_x(x, y) = y\cos(xy)$$

$$\Rightarrow \quad f_x\left(1, \frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = 0$$

$$f_y(x, y\_ = x\cos(xy)$$

$$\Rightarrow \quad f_y\left(1, \frac{\pi}{2}\right) = 1.\cos\frac{\pi}{2} = 0$$

$$f_{xx}(x, y) = -y^2\sin(x, y)$$

$\Rightarrow \quad f_{xx}\left(1, \frac{\pi}{2}\right) = \frac{-\pi^2}{4}\sin\frac{\pi}{2} = \frac{-\pi^2}{4}$

$f_{xy}(x, y) = -xy\sin(x, y)$

$\Rightarrow \quad f_{xy}\left(1, \frac{\pi}{2}\right) = \frac{-\pi}{2}\sin\frac{\pi}{2} = \frac{-\pi}{2}$

$f_{yy}(x, y) = -x^2\sin(x, y)$

$\Rightarrow \quad f_{yy}\left(1, \frac{\pi}{2}\right) = -1\sin\frac{\pi}{2} = -1$

From Taylor's series,

$$f(x, y) = f\left(1, \frac{\pi}{2}\right) + (x-1)f_x\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)f_y\left(1, \frac{\pi}{2}\right)$$

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1, \frac{\pi}{2}\right) + 2(x-1)\left(y - \frac{\pi}{2}\right)f_{xy}\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\left(1, \frac{\pi}{2}\right)\right]$$

Substituting the corresponding values in the above equation,

$$\sin(xy) = 1 + (x-1)(0) + \left(y - \frac{\pi}{2}\right)(0) + \frac{1}{2!}\left[(x-1)^2\left(\frac{-\pi^2}{4}\right) + 2(x-1)\left(y - \frac{\pi}{2}\right)\left(\frac{-\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2(-1)\right]$$

$$= 1 + \frac{1}{2}\left[(x-1)^2\left(\frac{-\pi}{4}\right)^2 - 2(x-1)\left(y - \frac{\pi}{2}\right)\frac{\pi}{2} - \left(y - \frac{\pi}{2}\right)^2\right]$$

$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2$$

$$\therefore \quad \sin(xy) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2.$$

3.8 MAXIMUM AND MINIMUM VALUES OF FUNCTIONS OF TWO VARIABLES, LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Q60. When a function f(x, y), with usual notations of partial differential coefficients, will have maximum, minimum and can't be decided?

Answer :

Given function is,

$f(x, y)$

Let (a, b) be the point at which maximum or minimum is determined. The condition for the function $f(x, y)$ to have maximum or minimum at a point (a, b) are $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Let,

$$\frac{\partial^2}{\partial x^2} f(a,b) = r$$

$$\frac{\partial^2}{\partial x \partial y} f(a,b) = t$$

$$\frac{\partial^2}{\partial y^2} f(a,b) = s$$

Then,

Case (i)

Function $f(a, b)$ is said to have maximum value if $rt - s^2 > 0$ and $r < 0$

Case (ii)

Function $f(a, b)$ is said to have minimum value if $rt - s^2 > 0$ and $r > 0$

Case (iii)

Function $f(a, b)$ does not have an extreme value i.e, neither maximum or minimum if $rt - s^2 < 0$.

Case (iv)

Function $f(a, b)$ fails to have maximum or minimum value and can't be decided if $rt - s^2 = 0$.

Q61. Explain Lagrange's method of undetermined multipliers.

Answer :

Let u be a function of three variables x, y, z given as,

$$u = f(x, y, z) \quad ... (1)$$

Let the variables are connected by a relation,

$$\phi(x, y, z) = 0 \quad ... (2)$$

Since, u is to have stationary value,

$$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du = 0 \quad ... (3)$$

Similarly equation (2) becomes,

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi = 0 \quad ... (4)$$

Multiplying equation (4) by λ,

$$\lambda\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = 0$$

$$\lambda\frac{\partial \phi}{\partial x}dx + \lambda\frac{\partial \phi}{\partial y} + \lambda\frac{\partial \phi}{\partial z}dz = 0 \quad ... (5)$$

Where

λ – Lagrange's multiplier

Adding equations (3) and (5),

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \lambda\frac{\partial \phi}{\partial x}dx + \lambda\frac{\partial \phi}{\partial y}dy + \lambda\frac{\partial \phi}{\partial z}dz = 0$$

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad ... (6)$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad ... (7)$$

$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad ... (8)$$

Equations (2), (6) (7) and (8) give the values of x, y, z and λ for which u is stationary.

Q62. Write the steps involved in Lagrange's multiplier method.

Ans :

Let, $u = f(x, y, z)$ be a function of three variables connected by a relation $\phi(x, y, z) = 0$

Step 1

The function is represented as,

$$F = u + \lambda\, \phi(x, y, z)$$

$$\Rightarrow \quad F = \phi(x, y, z) + \lambda\, \phi(x, y, z) \quad ... (1)$$

Where, λ – Lagrange's multiplier

Step 2

Differentiate equation (1) with respect to x, y and z and equate them to zero

i.e., $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Step 3

Solve the above three equations along with the relation $\phi(x, y, z)$ to obtain the values of x, y and z.

Step 4

Substitute the x, y, z values in the function u to obtain stationary value of u.

Q63. Examine the function f(x, y) = x³y²(12 – x – y) for extreme values.

Answer :

Given function is,

$$f(x, y) = x^3y^2(12 - x - y)$$

$$\Rightarrow \quad f = f(x, y) = 12x^3y^2 - x^4y^2 - x^3y^3 \qquad \text{... (1)}$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial f}{\partial x} = 36x^2y^2 - 4x^3y^2 - 3x^2y^3 \qquad \text{... (2)}$$

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial f}{\partial y} = 24x^3y - 2x^4y - 3x^3y^2 \qquad \text{... (3)}$$

Differentiating equation (2) partially with respect to 'x',

$$\frac{\partial^2 f}{\partial x^2} = 72xy^2 - 12x^2y^2 - 6xy^3 = r$$

Differentiating equation (2) partially with respect to 'y',

$$\frac{\partial^2 f}{\partial x \partial y} = 72x^2y - 8x^3y - 9x^2y^2 = s$$

Differentiating equation (3) partially with respect to 'y',

$$\frac{\partial^2 f}{\partial y^2} = 24x^3 - 2x^4 - 6x^3y = t$$

The conditions for maximum or minimum value i.e., extreme value are,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

From equation (2),

$$\frac{\partial f}{\partial x} = 0$$

$$\Rightarrow \quad 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$\Rightarrow \quad x^2y^2[36 - 4x - 3y] = 0$$

$$\Rightarrow \quad x = 0 \text{ or } y = 0$$

$$\text{or } 4x + 3y = 36$$

From equation (3),

$$\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \quad 24x^3y - 2x^4y - 3x^3y^2 = 0$$

$$\Rightarrow \quad x^3y[24 - 2x - 3y] = 0$$

$$\Rightarrow \quad x = 0 \text{ or } y = 0 \text{ or } 2x + 3y = 24$$

$\Rightarrow \quad (x, y) = (0, 0)$

$$\Rightarrow \quad \begin{aligned} 4x + 3y &= 36 \\ 2x + 3y &= 24 \\ - \quad - \quad &\; - \\ \hline 2x &= 12 \end{aligned}$$

$\Rightarrow \quad x = 6$

$2(6) + 3y = 24$

$\Rightarrow \quad 3y = 12$

$\Rightarrow \quad y = 4$

$\Rightarrow \quad (x, y) = (6, 4)$

$\therefore$ The stationary points are (0, 0), (6, 4)

Consider,

$$rt - s^2 = (72xy^2 - 12x^2y^2 - 6xy^3)\,(24x^3 - 2x^4 - 6x^3y) - (72x^2y - 8x^3y - 9x^2y^2)^2$$

$$rt - s^2|_{(0,\,0)} = \left(72(0)(0)^2 - 12(0)^2(0)^2 - 6(0)(0)^3\right)\,\left(24(0)^3 - 2(0)^4\right) - \left[72(0)^3(0) - 6(0)^2(0) - 8(0)^3(0) - 9(0)^2(0)^2\right]^2$$

$$= 0$$

$\therefore$ f does not have maximum or minimum value at (0, 0)

$$rt - s^2|_{(6,\,4)} = \left[72(6)(4)^2 - 12(6)^2(4)^2 - 6(6)(4)^3\right] \times \left[24(6)^3 - 2(6)^4 - 6(6)^3(4)\right] - \left[\left(72(6)^2(4) - 8(6)^3(4) - 9(6)^2(4)^2\right)\right]^2$$

$$= (6912 - 6912 - 2304)\,(5184 - 2592 - 5184) - [10368 - 6912 - 5184]^2$$

$$= 5971968 - (1728)^2$$

$$= 5971968 - 2985984$$

$$= 2985984 > 0$$

$\therefore$ f has maximum value at (6, 4)

Maximum value is obtained by substituting $x = 6$, $y = 4$ in equation (1)

$$f(x, y) = f(6, 4) = 12(6)^3(4)^2 - (6)^4(4)^2 - (6)^3(4)^3$$

$$= 41472 - 20736 - 13824$$

$$= 6912$$

$\therefore$ Maximum value = 6912.

Q64. Divide 24 into three points such that the continued product of the first, square of the second and cube of the third is maximum.

Answer :

Let x, y and z be three points.

$\Rightarrow \quad x + y + z = 24$ ... (1)

And the product of the first, square of the second and cube of the third point is maximum i.e., to maximize $(x \times y^2 \times z^3)$

Let, $\phi = xy^2z^3$ ... (2)

From equation (1),

$x = 24 - y - z$ ... (3)

Substituting the value of x from equation (3) in equation (2),

$$\phi = (24 - y - z)\, y^2z^3$$

$$\Rightarrow \quad \phi(y, z) = 24y^2z^3 - y^3z^3 - y^2z^4 \quad \text{... (4)}$$

Differentiating equation (4) partially with respect to 'y',

$$\frac{\partial \phi}{\partial y} = 48yz^3 - 3y^2z^3 - 2yz^4 \quad \text{... (5)}$$

Differentiating equation (4) partially with respect to 'z',

$$\frac{\partial \phi}{\partial z} = 72y^2z^2 - 3y^3z^2 - 4y^2z^3 \quad \text{... (6)}$$

For $\phi(y, z)$ to be maximum,

$$\frac{\partial \phi}{\partial y} = 0 \text{ and } \frac{\partial \phi}{\partial z} = 0$$

$\therefore$ Equation (5) becomes,

$$\frac{\partial \phi}{\partial y} = 48yz^3 - 3y^2z^3 - 2yz^4 = 0$$

$$\Rightarrow yz^3[48 - 3y - 2z] = 0$$

Since, $yz \neq 0$,

$$48 - 3y - 2z = 0$$

$$\Rightarrow \quad -3y - 2z = -48$$

$$\Rightarrow \quad 3y + 2z = 48 \quad \text{... (7)}$$

Equation (6) becomes,

$$\frac{\partial \phi}{\partial z} = 72y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0$$

$$\Rightarrow \quad y^2z^2[72 - 3y - 4z] = 0$$

Since, $yz \neq 0$

$$72 - 3y - 4z = 0$$

$$\Rightarrow \quad -3y - 4z = -72$$

$$\Rightarrow \quad 3y + 4z = 72 \quad \text{... (8)}$$

Solving equations (7) and (8),

$$y = 8 \text{ and } z = 12$$

Substituting the values of y and z in equation (1),

$$x + y + z = 24$$

$$\Rightarrow \quad x = 24 - y - z$$

$$\Rightarrow \quad x = 24 - 8 - 12$$

$$\Rightarrow \quad x = 24 - 20$$

$$\Rightarrow \quad x = 4$$

$\therefore$ $x = 4$, $y = 8$ and $z = 12$

$$r = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}[48yz^3 - 3y^2z^3 - 2yz^4]$$

$$\therefore \quad r = 48z^3 - 6yz^3 - 2z^4 \quad \text{... (9)}$$

$$s = \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial z}\right]$$

$$= \frac{\partial}{\partial y}[72y^2z^2 - 3y^3z^2 - 4y^2z^3]$$

$$\therefore \quad s = 144\, yz^2 - 9y^2z^2 - 8yz^3 \quad \text{... (10)}$$

$$t = \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial \phi}{\partial z}\right]$$

$$= \frac{\partial}{\partial z}[72y^2z^2 - 3y^3z^2 - 4y^2z^3]$$

$$\therefore \quad t = 144\, y^2z - 6y^3z - 12y^2z^2 \quad \text{... (11)}$$

At $y = 8$ and $z = 12$

Equation (9) becomes,

$$r = 48[12]^3 - 6[8][12]^3 - 2[12]^4$$

$$\Rightarrow r = 82944 - 82944 - 41472$$

$$\Rightarrow r = -41472 \quad \text{... (12)}$$

Equation (10) becomes,

$$s = 144[8]\,[12]^2 - 9[8]^2\,[12]^2 - 8[8]\,[12]^3$$

$$\Rightarrow \; s = 165888 - 82944 - 110592$$

$$\Rightarrow \; s = -27648 \quad \text{... (13)}$$

Equation (11) becomes,

$$t = 144[8]^2\,[12] - 6[8]^3\,[12] - 12[8]^2\,[12]^2$$

$$\Rightarrow \; t = 110592 - 36864 - 110592$$

$$\Rightarrow \; t = -36864 \quad \text{... (14)}$$

Consider,

$$rt - s^2 = [-41472 \times -36864] - [-27648]^2$$

$$= 764.41 \times 10^6 > 0$$

$\therefore$ $rt - s^2 > 0$ and $r < 0$

$\therefore$ $\phi = xy^2z^3$ is maximum when 24 is divided into $x = 4$, $y = 8$ and $z = 12$.

Q65. Find the stationary points of u(x, y) = sinx siny sin(x + y) where 0 < x < π, 0 < y < π and find the maximum u.

Answer :

Given function is,

$$u(x, y) = \sin x \sin y \sin(x + y)$$

Where, $0 < x < \pi$,

$$0 < y < \pi \quad \text{... (1)}$$

$$u = \sin x \sin y \sin(x + y)$$

Differentiating equation (1) partially with respect to x,

$$\frac{\partial u}{\partial x} = \cos x \sin y \sin(x+y) + \sin x \cos(x+y) \sin y$$
$$= \sin y [\cos x \sin(x+y) + \sin x \cos(x+y)]$$

$$\therefore \frac{\partial u}{\partial x} = \sin y \sin(2x+y)$$

[$\because \sin(A+B) = \cos A \sin B + \sin A \cos B$]

For stationary point,

$$\frac{\partial u}{\partial x} = 0$$

$\Rightarrow$ $\sin y \sin(2x+y) = 0$

$\Rightarrow$ $\sin y = 0$ or $\sin(2x+y) = 0$

[$\because 0 < x < \pi, 0 < y < \pi$]

$\Rightarrow$ $y = 0$ or $2x + y = \pi$ ... (2)

Differentiating equation (1) partially with respect to y,

$$\frac{\partial u}{\partial y} = \sin x \sin y \cos(x+y) + \sin x \cos y \sin(x+y)$$
$$= \sin x [\sin y \cos(x+y) + \sin(x+y) \cos y]$$

$$\therefore \frac{\partial u}{\partial y} = \sin x \sin(x+2y)$$

For stationary point,

$$\frac{\partial u}{\partial y} = 0$$

$\Rightarrow$ $\sin x \sin(x+2y) = 0$

$\Rightarrow$ $\sin x = 0$ or $\sin(x+2y) = 0$

$\Rightarrow$ $x = 0$ or $x + 2y = \pi$ ... (3)

Solving equations (2) and (3),

$\Rightarrow$ $x = \frac{\pi}{3}, \; y = \frac{\pi}{3}$

$\therefore$ Stationary point is $x = \frac{\pi}{3}, \; y = \frac{\pi}{3}$

$$r = \frac{\partial^2 u}{\partial x^2} = 2\sin y \cos(2x+y)$$

$$= 2\sin\frac{\pi}{3} \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$$

$$\therefore r = -\frac{2\sqrt{3}}{2} = -\sqrt{3}$$

$$s = \frac{\partial^2 u}{\partial x \partial y}$$
$$= \sin x \cos(x+2y) + \cos x \sin(x+2y)$$
$$= \sin(2x+2y)$$
$$= \sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right)$$

$$\therefore s = -\frac{\sqrt{3}}{2} \qquad \left[\because x = y = \frac{\pi}{3}\right]$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2 \sin x \cos(x+2y)$$

$$\therefore t = 2\sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) - \frac{2\sqrt{3}}{2} = -\sqrt{3}$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$$

$$\therefore rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$\therefore$ u will be maximum at $x = \frac{\pi}{3}, y = \frac{\pi}{3}$

$$\text{Maximum value} = \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\sin\left(\frac{2\pi}{3}\right)$$
$$= \frac{3}{4}\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

$$\therefore \text{Maximum value} = \frac{3\sqrt{3}}{8}$$

$$\therefore \text{Stationary point} = \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

Q66. Find the absolute maximum and minimum values for the function f(x, y) = $x^2 - y^2 - 2y$ in the closed region R : $x^2 + y^2 \leq 1$.

Answer : Dec.-16, Q14(b)

Given that,

$f(x, y) = x^2 - y^2 - 2y$ ... (1)

$R : x^2 + y^2 \leq 1$ ... (2)

Differentiating equation (1) partially with respect to x

$\frac{\partial f}{\partial x} = 2x$ ... (3)

Equating the value of $\frac{\partial f}{\partial x}$ to zero in order to determine the extreme point

$$\frac{\partial f}{\partial x} = 2x = 0$$

$\Rightarrow$ $2x = 0$

$\Rightarrow$ $x = 0$

Differentiating equation (2) partially with respect to y

$\frac{\partial f}{\partial y} = -2y - 2$ ... (4)

Equating the value of $\frac{\partial f}{\partial y}$ to zero in order to determine the extreme point

$$\frac{\partial f}{\partial y} = -2y - 2 = 0$$

$\Rightarrow \quad -2y-2=0$

$\Rightarrow \quad -2y=2$

$\Rightarrow \quad y=-1$

$\therefore$ The critical point $(x, y) = (0, -1)$

Differentiating equation (3) partially with respect to x

$$p = \frac{\partial^2 f}{\partial x^2} = 2$$

Differentiating equation (3) partially with respect to y

$$q = \frac{\partial^2 f}{\partial x \partial y} = 0$$

Differentiating equation (4) partially with respect to y

$$r = \frac{\partial^2 f}{\partial y^2} = -2$$

$$pr - q^2 = 2(-2) - 0^2$$
$$= -4 < 0$$

As $pr - q^2 < 0$, the function $f(x, y)$ does not have extreme value.

As $\frac{\partial^2 f}{\partial x^2} > 0$, the function has relative minimum value.

The relative minimum value at $(0, -1)$ is given as,

$$f(0, -1) = 0^2 - (-1)^2 - 2(-1)$$
$$= 0 - 1 + 2 = 1$$

From equations (2);

$$x^2 = 1 - y^2 \qquad \text{... (5)}$$

Substituting equation (5) in equation (1)

$\Rightarrow \quad g(y) = (1-y^2) - y^2 - 2y$

$\Rightarrow \quad g(y) = 1 - 2y^2 - 2y \qquad \text{... (6)}$

Differentiating equation (6) partially with respect to y

$$\frac{\partial g}{\partial y} = 0 - 4y - 2$$

$\Rightarrow \quad \frac{\partial g}{\partial y} = -4y - 2 \qquad \text{... (7)}$

Equating $\frac{\partial g}{\partial y}$ to zero;

$\Rightarrow \quad -4y - 2 = 0$

$\Rightarrow \quad -4y = 2$

$\Rightarrow \quad y = \frac{-1}{2}$

Substituting $y = \frac{-1}{2}$ in equation (5)

$$x^2 = 1 - \left(\frac{-1}{4}\right)^2$$

$\Rightarrow \quad x^2 = 1 - \frac{1}{4}$

$\Rightarrow \quad x^2 = \frac{3}{4}$

$\Rightarrow \quad x = \pm\frac{\sqrt{3}}{2}$

$\therefore$ The critical region on the boudary is $(x, y) = \left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

Differentiating equation (7) partially with respect to y

$$\frac{\partial^2 g}{\partial y^2} = -4 < 0$$

As $\frac{\partial^2 g}{\partial y^2} < 0$, the function $f(x, y)$ has relative maximum value.

$$f\left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{2}\right) = \left(\pm\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{-1}{2}\right)^2 - 2\left(\frac{-1}{2}\right)$$
$$= \frac{3}{4} - \frac{1}{4} + 1 = \frac{3}{2}$$

$\therefore$ The given function has relative minimum value as 1 at $(0, -1)$ and relative maximum value as $\frac{3}{2}$ at $\left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

Q67. Find the maximum value of $x^2y^3z^4$ subject to the condition $2x + 3y + 4z = a$.

Answer :

Consider the given functions as,

$$\phi(x, y, z) = 2x + 3y + 4z - a = 0 \qquad \text{... (1)}$$

And $u(x, y, z) = x^2y^3z^4 \qquad \text{... (2)}$

According to Lagrange's function,

$$F(x, y, z) = u(x, y, z) + \lambda\phi(x, y, z) \qquad \text{... (3)}$$

Where, λ = Lagrange's multiplier.

Substituting equations (1) and (2) in equation (3),

$$F(x, y, z) = x^2y^3z^4 + \lambda(2x + 3y + 4z - a) \qquad \text{... (4)}$$

Differentiating equation (4) partially with respect to 'x',

$$\frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda$$

Differentiating equation (4) partially with respect to 'y',

$$\frac{\partial F}{\partial y} = 3y^2x^2z^4 + 3\lambda$$

Differentiating equation (4) partially with respect to 'z',

$$\frac{\partial F}{\partial z} = 4z^3x^2y^3 + 4\lambda$$

For maximum or minimum value, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Consider,

$$\frac{\partial F}{\partial x} = 0$$

$\Rightarrow \quad 2xy^3z^4 + 2\lambda = 0$

$\Rightarrow \quad 2(xy^3z^4 + \lambda) = 0$

$\Rightarrow \quad xy^3z^4 + \lambda = 0$

Multiplying and dividing by 'x',

$$\Rightarrow \frac{x}{x}.xy^3z^4 + \lambda = 0$$

$$\Rightarrow \frac{1}{x}.x^2y^3z^4 = -\lambda$$

$$\Rightarrow \frac{u}{x} = -\lambda \qquad [\because \text{ From equation (2)}]$$

$$\therefore x = \frac{-u}{\lambda}$$

$$\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow 3y^2x^2z^4 + 3\lambda = 0$$

$$\Rightarrow 3(y^2x^2z^4 + \lambda) = 0$$

$$\Rightarrow y^2x^2z^4 + \lambda = 0$$

Multiplying and dividing by 'y',

$$\Rightarrow \frac{y}{y}.y^2x^2z^4 + \lambda = 0$$

$$\Rightarrow \frac{y^3x^2z^4}{y} = -\lambda$$

$$\Rightarrow \frac{u}{y} = -\lambda \qquad [\because \text{ From equation (2)}]$$

$$\therefore y = \frac{-u}{\lambda}$$

$$\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow 4z^3x^2y^3 + 4\lambda = 0$$

$$\Rightarrow 4(z^3x^2y^3 + \lambda) = 0$$

$$\Rightarrow z^3x^2y^3 + \lambda = 0$$

Multiplying and dividing by 'z',

$$\Rightarrow \frac{z}{z}.z^3x^2y^3 + \lambda = 0$$

$$\Rightarrow \frac{1}{z}.x^2y^3z^4 = -\lambda$$

$$\Rightarrow \frac{u}{z} = -\lambda \qquad [\because \text{ From equation (2)}]$$

$$\therefore z = \frac{-u}{\lambda}$$

$$x = y = z = \frac{-u}{\lambda} \qquad \dots (5)$$

Substituting the values of x, y and z in equation (1),

$$\phi(x, y, z) = 2x + 3y + 4z - a = 0$$

$$\Rightarrow 2\left(\frac{-u}{\lambda}\right) + 3\left(\frac{-u}{\lambda}\right) + 4\left(\frac{-u}{\lambda}\right) - a = 0$$

$$\Rightarrow \frac{-u}{\lambda}[2+3+4] = a$$

$$\Rightarrow \frac{-u}{\lambda}[9] = a$$

$$\Rightarrow -9u = a\lambda$$

$$\therefore \lambda = \frac{-9u}{a}$$

Substituting the value of 'λ' in equation (5),

$$\Rightarrow x = y = z = \frac{-u}{\lambda}$$

$$= \frac{-u}{\left(\frac{-9u}{a}\right)}$$

$$= \frac{-au}{-9u} = \frac{a}{9}$$

$$\therefore x = y = z = \frac{a}{9}$$

The maximum value is given as,

$$= \left(\frac{a}{9}\right)^2\left(\frac{a}{9}\right)^3\left(\frac{a}{9}\right)^4 = \left(\frac{a}{9}\right)^{2+3+4} = \left(\frac{a}{9}\right)^9$$

$\therefore$ The maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$.

Q68. Find the maximum and minimum values of $x + y + z$ subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$.

Answer :

Given function is,

$$f(x, y, z) = x + y + z$$

It is basically a constrained extremum problem, where a function 'f' is subjected to the constraint.

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$$

Substituting both the function and its constraint in the following auxiliary function.

$$F(x, y, z) = x + y + z + \lambda\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right] \qquad \dots (1)$$

Differentiating equation (1) partially with respect to x, y, z and then equating the results to zero.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x+y+z) + \lambda \frac{\partial}{\partial x}\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right]$$

$$= 1 + \left[\frac{-\lambda}{x^2}\right]$$

$$= 1 - \frac{\lambda}{x^2}$$

$$= \frac{x^2-\lambda}{x^2}$$

$$\therefore \quad \frac{\partial F}{\partial x} = x^2 - \lambda \qquad \text{... (2)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x+y+z) + \lambda \frac{\partial}{\partial y}\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right]$$

$$= [1] + \left[\frac{-\lambda}{y^2}\right]$$

$$= \frac{y^2-\lambda}{y^2}$$

$$\therefore \quad \frac{\partial F}{\partial y} = y^2 - \lambda \qquad \text{... (3)}$$

Similarly,

$$\frac{\partial F}{\partial z} = 1 - \frac{\lambda}{z^2} = \frac{z^2-\lambda}{z^2}$$

$$\therefore \quad \frac{\partial F}{\partial z} = \frac{z^2-\lambda}{z^2} \qquad \text{... (4)}$$

Equating equations (2), (3) and (4) to zero,

$$x^2 - \lambda = 0, \quad y^2 - \lambda = 0, \quad z^2 - \lambda = 0$$

$$x^2 = \lambda, \quad y^2 = \lambda, \quad z^2 = \lambda$$

$$x = \pm\sqrt{\lambda}, \quad y = \pm\sqrt{\lambda} \text{ and } z = \pm\sqrt{\lambda}$$

Substituting the values of x, y, z in the given constraint,

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}} = 1$$

$$\Rightarrow \quad \frac{1+1+1}{\sqrt{\lambda}} = 1$$

$$\Rightarrow \quad \frac{3}{\sqrt{\lambda}} = 1$$

$$\Rightarrow \quad 3 = \sqrt{\lambda}$$

Squaring on both sides,

$$\Rightarrow \quad \lambda = 3^2 = 9$$

$$\therefore \quad \lambda = 9$$

Substituting the value of λ in x, y and z,

$$x = \pm 3, \; y = \pm 3, \; z = \pm 3$$

$\therefore$ The extreme i.e., maximum and minimum values are + 9 and – 9.

Q69. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Answer :

Let, the coordinates of the rectangular parallelopiped be (x, y, z).

The volume of the rectangular parallelopiped,

$$V = 8xyz \qquad \text{... (1)}$$

Maximum volume is to be calculated such that it is inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Let,

$$F = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \qquad \text{... (2)}$$

The condition for extreme value is given by,

$$dV = yz\,dx + zx\,dy + xy\,dz = 0 \qquad \text{... (3)}$$

Differentiating equation (2),

$$dF = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0$$

$$\Rightarrow dF = \frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \qquad \text{... (4)}$$

Multiplying equation (4) by λ and adding equations (3) and (4),

$$(yz\,dx + zx\,dy + xy\,dz) + \left(\frac{\lambda x}{a^2}dx + \frac{\lambda y}{b^2}dy + \frac{\lambda z}{c^2}dz\right) = 0$$

$$\left(yz + \frac{\lambda x}{a^2}\right)dx + \left(xz + \frac{\lambda y}{b^2}\right)dy + \left(xy + \frac{\lambda z}{c^2}\right)dz = 0$$

Equating the coefficients of dx, dy, dz to zero,

$$\Rightarrow \quad \left(yz + \frac{\lambda x}{a^2}\right) = 0 \qquad \text{... (5)}$$

$$\Rightarrow \quad \left(xz + \frac{\lambda y}{b^2}\right) = 0 \qquad \text{... (6)}$$

$$\Rightarrow \quad \left(xy + \frac{\lambda z}{c^2}\right) = 0 \qquad \text{... (7)}$$

From equations (5), (6) and (7),

$$\Rightarrow \quad \lambda = \frac{-yza^2}{x} = \frac{-zxb^2}{y} = \frac{-xyc^2}{z}$$

$$\Rightarrow \frac{a^2yz}{x} = \frac{b^2zx}{y} = \frac{c^2xy}{z} \quad \text{... (8)}$$

Dividing throughout equation (8) by xyz,

$$\Rightarrow \frac{a^2yz}{x^2yz} = \frac{b^2zx}{xy^2z} = \frac{c^2xy}{xyz^2}$$

$$\Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Substituting, $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$ in equation (2),

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \qquad \left[\because \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{x^2}{a^2}\right]$$

$$\Rightarrow \frac{3x^2}{a^2} = 1$$

$$\Rightarrow x^2 = \frac{a^2}{3}$$

$$\Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly, $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$.

Thus, the stationary points are $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

Differentiating equation (2) partially with respect to 'x' and keeping y as constant,

$$\frac{1}{a^2}(2x) + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = \frac{-2x}{a^2}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-2x}{a^2} \times \frac{c^2}{2z}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-xc^2}{a^2z} \quad \text{... (9)}$$

Similarly, differentiating equation (1) partially with respect to 'x' and keeping y as constant,

$$\frac{\partial V}{\partial x} = 8yz + 8xy\frac{\partial z}{\partial x} \quad \text{... (10)}$$

Substituting equation (9) in equation (10),

$$= 8yz + 8xy\left(\frac{-c^2x}{a^2z}\right)$$

$$\therefore \quad \frac{\partial V}{\partial x} = 8yz - \frac{8x^2yc^2}{a^2z}$$

Again differentiating equation (5) partially with respect to 'x' keeping 'y' as constant,

$$\frac{\partial^2 V}{\partial x^2} = 8y\left(\frac{\partial z}{\partial x}\right) - \frac{8yc^2}{a^2}\frac{\partial}{\partial x}\left(\frac{x^2}{z}\right)$$

$$\therefore \quad \frac{\partial^2 V}{dx^2} = 8y\left(\frac{-c^2x}{a^2z}\right) - \frac{16c^2xy}{a^2z} - \frac{8c^2x^2y}{a^2z}.\frac{c^2x}{a^2z}$$

Since, the value of $\frac{\partial^2 V}{\partial x^2}$ is negative,

The maximum value is obtained at the stationary point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

From equation (1), $V = 8xyz$

$$\therefore \text{ Maximum value, } V_{max} = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}}$$

$$V_{max} = \frac{8abc}{3\sqrt{3}}.$$

Q70. Find the shortest and longest distances from the point (1, 2, – 1) to the sphere $x^2 + y^2 + z^2 = 24$.

Answer :

Given equation of sphere is,

$$x^2 + y^2 + z^2 = 24 \quad \text{... (1)}$$

Point (1, 2, – 1)

Let (x, y, z) be the required point on the sphere

Let d be the distance between the points (x, y, z) and (1, 2, – 1)

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

$$\Rightarrow \quad d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$$

Let $f = (x-1)^2 + (y-2)^2 + (z+1)^2$

$$g = x^2 + y^2 + z^2 - 24$$

The auxiliary function is given by

$$F = f + \lambda g$$

Where λ is Lagrange's multiplier

i.e., $F = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$

The necessary conditions for maximum or minimum values are,

$$\frac{\partial F}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial y} = 0$$

i.e., $2(x-1) + 2\lambda x = 0$

$$\Rightarrow \quad (x-1) = -\lambda x$$

$$\Rightarrow \quad \frac{x-1}{x} = -\lambda$$

$$\Rightarrow \quad \frac{-x+1}{x} = \lambda$$

$\Rightarrow \quad -1+\frac{1}{x}=\lambda$

$\Rightarrow \quad \frac{1}{x}=\lambda+1$

$\Rightarrow \quad x=\frac{1}{\lambda+1}$ ... (2)

$\frac{\partial F}{\partial y}=0$

i.e., $2(y-2)+2\lambda y=0$

$\Rightarrow \quad (y-2)=-\lambda y$

$\Rightarrow \quad \frac{y-2}{y}=-\lambda$

$\Rightarrow \quad \frac{-y+2}{y}=\lambda$

$\Rightarrow \quad -1+\frac{2}{y}=\lambda$

$\Rightarrow \quad \frac{2}{y}=\lambda+1$

$\Rightarrow \quad y=\frac{2}{\lambda+1}$ ... (3)

$\frac{\partial F}{\partial z}=0$

i.e., $2(z+1)+2\lambda z=0$

$\Rightarrow \quad z+1=-\lambda z$

$\Rightarrow \quad \frac{z+1}{z}=-\lambda$

$\Rightarrow \quad -1-\frac{1}{z}=\lambda$

$\Rightarrow \quad \frac{-1}{z}=\lambda+1$

$\Rightarrow \quad z=\frac{-1}{\lambda+1}$ ... (4)

Substituting the values of x, y and z in equation (1),

$\Rightarrow \quad \left(\frac{1}{\lambda+1}\right)^2+\left(\frac{2}{\lambda+1}\right)^2+\left(\frac{-1}{\lambda+1}\right)^2=24$

$\Rightarrow \quad \frac{1}{(\lambda+1)^2}+\frac{4}{(\lambda+1)^2}+\frac{1}{(\lambda+1)^2}=24$

$\Rightarrow \quad \frac{6}{(\lambda+1)^2}=24$

$\Rightarrow \quad (\lambda+1)^2=\frac{6}{24}$

$\Rightarrow \quad (\lambda+1)^2=\frac{1}{4}$

$\Rightarrow \quad (\lambda+1)=\pm\frac{1}{2}$

$\lambda+1=\frac{1}{2} \quad ; \quad \lambda+1=\frac{-1}{2}$

$\lambda=\frac{1}{2}-1 \quad ; \quad \lambda=\frac{-1}{2}-1$

$\lambda=\frac{-1}{2} \quad ; \quad \lambda=\frac{-3}{2}$

Substituting $\lambda=\frac{-1}{2}$ in equations (2), (3) and (4)

$x=\frac{1}{\lambda+1}=\frac{1}{-\frac{1}{2}+1}=\frac{1}{\frac{1}{2}}=2$

$y=\frac{2}{\lambda+1}=\frac{2}{\frac{-1}{2}+1}=\frac{2}{\frac{1}{2}}=4$

$z=\frac{-1}{\lambda+1}=\frac{-1}{\frac{-1}{2}+1}=\frac{-1}{\frac{1}{2}}=-2$

$\therefore \quad (x, y, z)=(2, 4, -2)$

Substituting $\lambda=\frac{-3}{2}$ in equations 2, 3, and 4

$x=\frac{1}{\lambda+1}=\frac{1}{\frac{-3}{2}+1}=\frac{1}{\frac{-1}{2}}=-2$

$y=\frac{2}{\lambda+1}=\frac{2}{\frac{-3}{2}+1}=\frac{2}{\frac{-1}{2}}=-4$

$z=\frac{-1}{\lambda+1}=\frac{-1}{\frac{-3}{2}+1}=\frac{-1}{\frac{-1}{2}}=2$

$\therefore \quad (x, y, z)=(2, 4, -2)$

At the point (2, 4, – 2)

The distance d at (2, 4, – 2) is,

$d=\sqrt{(2-1)^2+(4-2)^2+(-2+1)^2}$

$=\sqrt{(1)^2+(2)^2+(-1)^2}$

$=\sqrt{1+4+1}$

$\therefore \; d=\sqrt{6}$ units ... (5)

At the point (– 2, – 4, 2)

The distance d at (– 2, – 4, 2) is,

$d=\sqrt{(-2-1)^2+(-4-2)^2+(2+1)^2}$

$=\sqrt{(-3)^2+(-6)^2+(3)^2}$

$=\sqrt{9+36+9}$

$=\sqrt{54}$

$\therefore \; d=3\sqrt{6}$ units ... (6)

It can be seen from equations (5) and (6) that shortest distance is $\sqrt{6}$ units and longest distance is, $3\sqrt{6}$ units

$\therefore$ Shortest distance = $\sqrt{6}$ units

Longest distance = $3\sqrt{6}$ units.

UNIT 4

MULTIVARIABLE CALCULUS (INTEGRATION)

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Evaluate: $\int_0^5 \int_0^2 (x^2+y^2)dxdy$ **.**

Answer :

Given integral is,

$$\int_0^5 \int_0^2 (x^2+y^2)\,dx\,dy$$

The above integral can be evaluated as,

$$\int_0^5 \int_0^2 (x^2+y^2)\,dx\,dy = \int_0^5 \left[\int_0^2 (x^2+y^2)\,dx\right] dy$$

$$= \int_0^5 \left[\frac{x^3}{3}+xy^2\right]_0^2 dy$$

$$= \int_0^5 \left[\frac{8}{3}+2y^2\right] dy = \int_0^5 \left(\frac{8}{3}+2y^2\right) dy$$

$$= \frac{8}{3}\int_0^5 dy + 2\int_0^5 y^2\,dy = \frac{8}{3}[y]_0^5 + 2\left[\frac{y^3}{3}\right]_0^5$$

$$= \frac{8}{3}[5-0] + \frac{2}{3}[125-0]$$

$$= \frac{40}{3}+\frac{250}{3} = \frac{290}{3}$$

$$\therefore \quad \int_0^5 \int_0^2 (x^2+y^2)\,dx\,dy = \frac{290}{3}.$$

Q2. Evaluate $\int_0^1 \int_0^x e^{x+y}dydx$

Answer :

Given integral is,

$$\int_0^1 \int_0^x e^{x+y}\,dydx$$

The above integral can be evaluated as,

$$\int_0^1\int_0^x e^{x+y}\,dydx = \int_0^1\int_0^x e^x.e^y\,dydx$$

$$= \int_0^1 e^x\left[\int_0^x e^y\,dy\right]dx$$

$$= \int_0^1 e^x\left[e^y\right]_0^x dx$$

$$= \int_0^1 e^x\left[e^x - e^0\right]dx$$

$$= \int_0^1 e^x\left[e^x - 1\right]dx$$

$$= \int_0^1 (e^{2x} - e^x)dx$$

$$= \int_0^1 e^{2x}dx - \int_0^1 e^x dx$$

$$= \left[\frac{e^{2x}}{2}\right]_0^1 - \left[e^x\right]_0^1$$

$$= \left[\frac{e^2 - e^0}{2}\right] - [e^1 - e^0]$$

$$= \frac{e^2-1}{2} - (e-1) = \frac{e^2-1-2e+2}{2}$$

$$= \frac{e^2-2e+1}{2} = \frac{(e-1)^2}{2}$$

$$\therefore \quad \int_0^1\int_0^x e^{x+y}\,dydx = \frac{(e-1)^2}{2}.$$

Q3. Evaluate $\int_1^b\int_1^a \frac{dxdy}{xy}$

Answer :

Given integral is,

$$\int_1^b\int_1^a \frac{dx\,dy}{xy}$$

The above integral can be evaluated as,

$$\int_1^b\int_1^a \frac{dx\,dy}{xy} = \int_1^b\left[\int_1^a \frac{1}{xy}dx\right]dy$$

$$= \int_1^b\left[\frac{1}{y}[\log x]_1^a\right]dy$$

$$= \int_1^b\left[\frac{1}{y}[\log a - \log 1]\right]dy$$

$$= \int_1^b \frac{1}{y}(\log a)dy$$

$$= \log a\int_1^b \frac{1}{y}\,dy$$

$$= \log a[\log y]_1^b$$

$$= \log a[\log b - 0] \qquad [\because \log 1 = 0]$$

$$= \log a \log b$$

$$\therefore \quad \int_1^b\int_1^a \frac{dx\,dy}{xy} = \log a \log b$$

Q4. Evaluate $\int_2^3\int_1^2 \frac{1}{xy}dxdy$

Answer :

Given integral is,

$$\int_2^3\int_1^2 \frac{1}{xy}dx\,dy$$

The above integral can be evaluated as,

$$\int_2^3\int_1^2 \frac{1}{xy}dx\,dy = \int_2^3\left[\int_1^2 \frac{1}{xy}dx\right]dy$$

$$= \int_2^3 \frac{1}{y}[\log x]_1^2\,dy$$

$$= \int_2^3 (\log 2 - \log 1)\,\frac{1}{y}\,dy$$

$$= (\log 2 - \log 1)\int_2^3 \frac{dy}{y}$$

$$= \log 2\,[\log y]_2^3$$

$$= \log 2[\log 3 - \log 2]$$

$$= \log 2.\log\frac{3}{2}$$

$$\therefore \quad \int_2^3\int_1^2 \frac{1}{xy}dx\,dy = \log 2.\log\frac{3}{2}$$

Q5. Evaluate $\int_0^4 \int_0^{x^2} e^{\frac{y}{x}} dy\, dx$.

Answer :

Given integral is,

$$\int_0^4 \int_0^{x^2} e^{\frac{y}{x}} dy\, dx$$

The above integral can be evaluated as,

$$\int_0^4 \int_0^{x^2} e^{\frac{y}{x}} dy\, dx = \int_0^4 dx \int_0^{x^2} e^{\frac{y}{x}} dy$$

$$= \int_0^4 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^{x^2} dx$$

$$= \int_0^4 \left[x\, e^{\frac{y}{x}}\right]_0^{x^2} dx$$

$$= \int_0^4 \left[x\left(e^{\frac{x^2}{x}} - e^0\right)\right] dx$$

$$= \int_0^4 [x(e^x - 1)]dx$$

$$= \int_0^4 (xe^x - x)dx$$

$$= \int_0^4 xe^x dx - \int_0^4 x\, dx$$

$$= [xe^x - e^x]_0^4 - \left[\frac{x^2}{2}\right]_0^4$$

$$= [(4e^4 - e^4) - (0 - e^0)] - \left[\frac{16}{2} - 0\right]$$

$$= e^4(4 - 1) + 1 - 8$$

$$= 3e^4 - 7$$

$$\therefore \quad \int_0^4 \int_0^{x^2} e^{\frac{y}{x}} dy\, dx = 3e^4 - 7.$$

Q6. Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta$.

Answer :

Given integral is,

$$\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta$$

The above integral can be evaluated as,

$$\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta = \int_0^{\pi} \left[\int_0^{\sin\theta} r\, dr\right] d\theta$$

$$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

$$= \int_0^{\pi} \left[\frac{\sin^2\theta}{2} - 0\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \frac{(1 - \cos 2\theta)}{2} d\theta$$

$$= \frac{1}{4}\left[\int_0^{\pi} d\theta - \int_0^{\pi} \cos 2\theta\, d\theta\right]$$

$$= \frac{1}{4}\left[[\theta]_0^{\pi} - \left[\frac{\sin 2\theta}{2}\right]_0^{\pi}\right]$$

$$= \frac{1}{4}\left[(\pi - 0) - \left(\frac{\sin 2\pi}{2} - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{4}[\pi - (0 - 0)]$$

$$= \frac{\pi}{4}$$

$$\therefore \quad \int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta = \frac{\pi}{4}.$$

Q7. Find the value of $\int_0^{\infty} \int_0^{y} \left(\frac{e^{-y}}{y}\right) dx\, dy$

Answer :

Given integral is,

$$\int_0^{\infty} \int_0^{y} \left(\frac{e^{-y}}{y}\right) dx\, dy$$

The above integral can be written as,

$$\int_0^{\infty} \int_0^{y} \left(\frac{e^{-y}}{y}\right) dx\, dy = \int_0^{\infty} \left[\int_0^{y} dx\right] \frac{e^{-y}}{y} dy$$

$$= \int_0^{\infty} [x]_0^y \frac{e^{-y}}{y} dy$$

$$= \int_0^{\infty} [y - 0] \frac{e^{-y}}{y} dy$$

$$= \int_0^\infty y.\frac{e^{-y}}{y}\,dy$$

$$= \int_0^\infty e^{-y}dy$$

$$= [-e^{-y}]_0^\infty$$

$$= [-e^{-\infty} + e^{-0}]$$

$$= [0 + 1]$$

$$= 1$$

$$\therefore \int_0^\infty \int_0^y \left(\frac{e^{-y}}{y}\right) dx\,dy = 1.$$

Q8. Evaluate $\int_0^\pi \int_0^a r dr d\theta$.

Answer :

Given integral is,

$$\int_0^\pi \int_0^a r dr d\theta$$

The above integral can be evaluated as,

$$\int_0^\pi \int_0^a r dr d\theta = \int_0^\pi \left[\int_0^a r dr\right] d\theta$$

$$= \int_0^\pi \left[\frac{r^2}{2}\right]_0^a d\theta$$

$$= \int_0^\pi \left[\frac{a^2}{2} - 0\right] d\theta$$

$$= \frac{a^2}{2}\int_0^\pi d\theta$$

$$= \frac{a^2}{2}[\theta]_0^\pi$$

$$= \frac{a^2}{2}[\pi - 0]$$

$$= \frac{\pi a^2}{2}$$

$$\therefore \int_0^\pi \int_0^a r dr d\theta = \frac{\pi a^2}{2}.$$

Q9. Evaluate $\int_0^2 \int_0^\pi r\sin^2\theta d\theta dr$

Answer :

Given integral is,

$$\int_0^2 \int_0^\pi r\sin^2\theta d\theta dr$$

The above integral can be evaluated as,

$$\int_0^2 \int_0^\pi r\sin^2\theta d\theta dr = \int_0^2 r dr \int_0^\pi \sin^2\theta d\theta$$

$$= \int_0^2 r dr \int_0^\pi \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\int_0^2 r \int_0^\pi \left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi dr$$

$$= \frac{1}{2}\int_0^2 r\left[\left(\pi - \sin\frac{2\pi}{2}\right) - (0-0)\right] dr$$

$$= \frac{1}{2}\int_0^2 r(\pi - 0)\,dr$$

$$= \frac{\pi}{2}\int_0^2 r dr$$

$$= \frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^2$$

$$= \frac{\pi}{2}\left[\frac{4}{2} - 0\right]$$

$$= \pi$$

$$\therefore \int_0^2 \int_0^\pi r\sin^2\theta d\theta dr = \pi.$$

Q10. Find the limits of integration in the double integral $\iint_R f(x, y)\,dx\,dy$ where R is in the first quadrant and bounded x = 1, y = 0, $y^2 = 4x$.

Answer :

Given integral is,

$$\iint_R f(x,y)\,dx\,dy$$

The region R is bounded by the lines $x = 1$, $y = 0$, $y^2 = \phi x$

For $x = 1$,

$y^2 = 4(1)$

$\Rightarrow$ $y^2 = 4$

$\Rightarrow$ $y = \sqrt{4}$

$\Rightarrow$ $y = 2$

$\therefore$ x varies from 0 to 1

y varies from 0 to 2.

Q11. Find the area bounded by the lines x = 0, y = 1 andy = x, using double integration.

Answer :

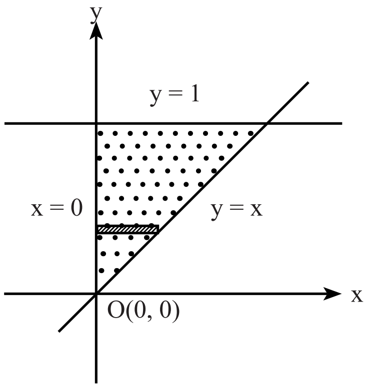
Given lines are,

$x = 0$

$y = 1$

$y = x$

The region of integration is illustrated in figure



Figure

$x = 0 \Rightarrow y = 0$

$\therefore$ y varies from 0 to 1

$y = x \Rightarrow x = y$

$\therefore$ y varies from 0 to y

Area is given as,

$$\text{Area} = \int_0^1 \int_0^y dx\, dy$$

$$= \int_0^1 [x]_0^y\, dy$$

$$= \int_0^1 (y - 0) dy$$

$$= \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

$\therefore$ Area $= \frac{1}{2}$ square units.

Q12. Find the area bounded by the lines x = 0, y = 1, x = 1 and y = 0.

Answer :

Given lines are,

$x = 0$

$y = 1$

$x = 1$

$y = 0$

Limits for x

$x = 0$

$x = 1$

$\therefore$ x varies from 0 to 1

Limits for y

$y = 1$

$y = 0$

$\therefore$ y varies from 0 to 1

$\therefore$ The area bounded by the lines is given as,

$$\text{Area} = \int_0^1 \int_0^1 dy\, dx = \int_0^1 \left[\int_0^1 dy\right] dx$$

$$= \int_0^1 [y]_0^1 dx$$

$$= \int_0^1 [1 - 0] dx$$

$$= \int_0^1 dx$$

$$= [x]_0^1$$

$$= [1 - 0]$$

$$= 1$$

$\therefore$ $\int_0^1 \int_0^1 dy\, dx = 1$ square units.

Q13. Find the limits of integration after changing the order of integration of $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$.

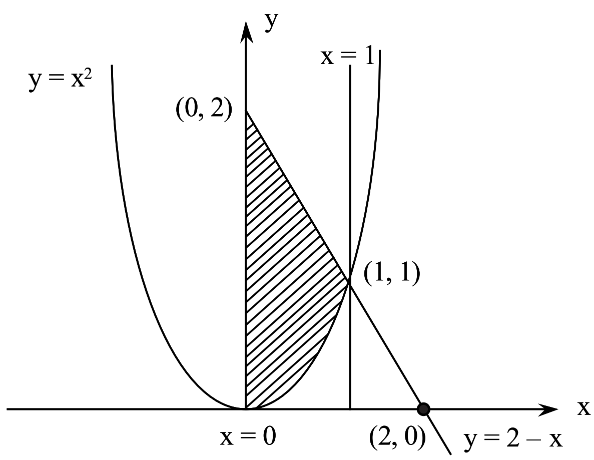
Answer :

Given integral is,

$$\int_0^1 \int_{x^2}^{2-x} xydydx \quad \text{... (1)}$$

It can be observed that for a fixed value of x, y varies from x^2 to $2 - x$ and then x is varied from 0 to 1.

Figure below illustrates the lines, $y = x^2$, $y = 2 - x$ and $x = 0$, $x = 1$.



Figure

Change of Order

$y = 2 - x$ ... (1)

But,

$y = x^2$ ... (2)

$\therefore$ $x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2, x = 1$

Substituting x values in equation (2),

If $x = -2 \Rightarrow y = 4$

If $x = 1 \Rightarrow y = 1$

$\therefore$ Integration limits are,

$x = -2$ to 1

$y = 4$ to 1.

Q14. Change the order of integration in $\int_0^1 \int_0^y f(x, y)\,dx\,dy$.

Answer :

Given integral is,

$$\int_0^1 \int_0^y f(x, y)\,dx\,dy$$

$\therefore$ y varies from 0 to 1

$\therefore$ x varies from 0 to y

The region of integration is bounded by the lines $y = 0$, $y = 1$ and $x = 0$, $x = y$

Figure 1 represents the region of integration.

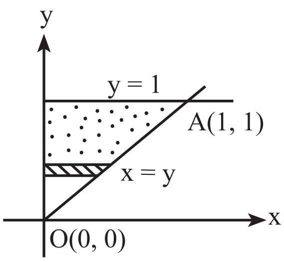


Figure (1)

The point of intersection is $A(1, 1)$

A vertical strip is drawn parallel to y-axis that starts from the line $x = y$ and ends on the line $y = 1$ as shown in figure 2.

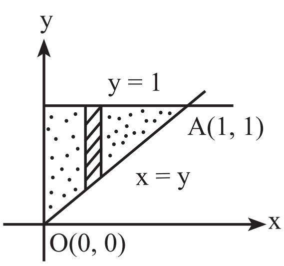


Figure (2)

$\Rightarrow$ $y = x$

$\therefore$ y varies from 0 to 1

x varies from 0 to 1

$$\therefore \int_0^1 \int_0^y f(x,y)\,dx\,dy = \int_0^1 \int_x^1 f(x,y)\,dy\,dx$$

Q15. Express $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dxdy$ into polar coordinates.

Answer :

Given integral is,

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$$

The region of integration is bounded by the planes $y = 0$, $y = a$, $x = y$, $x = a$

Let, $x = r\cos\theta$

$y = r\sin\theta$

and $dx\,dy = r dr\,d\theta$

If $x = a$, then

$a = r\cos\theta$

$\Rightarrow$ $r = \dfrac{a}{\cos\theta}$

$\Rightarrow$ $r = a\sec\theta$

If $x = 0$, then

$0 = r\cos\theta$

$\Rightarrow$ $r = 0$

$\therefore$ r varies from 0 to $a\sec\theta$

If $y = 0$, then

$0 = r\sin\theta$

$\Rightarrow$ $\theta = 0$

Since $x = y$

$r\cos\theta = r\sin\theta$

$\cos\theta = \sin\theta$

$\Rightarrow$ $\theta = \dfrac{\pi}{4}$

$\therefore$ θ varies from 0 to $\dfrac{\pi}{4}$

$$\therefore \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=a\sec\theta} \left(\frac{r^2\cos^2\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}\, r\,dr d\theta\right)$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r^3\cos^2\theta}{\sqrt{r^2(1)}}\,dr\,d\theta$$

$$\therefore \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dxdy = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} r^2\cos^2\theta\,dr\,d\theta$$

Q16. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dydx$ by changing into polar coordinates.

Answer :

Given integral is,

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dx\,dy \quad \dots (1)$$

Equation (1) can be written in standard form as,

$$\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} (x^2+y^2)dy\;dx$$

Let, $x = r\cos\theta$ and $y = r\sin\theta$

$dxdy = rdrd\theta$

$x^2 + y^2 = a^2$

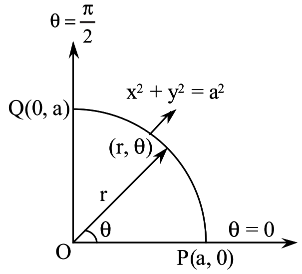
$\Rightarrow \quad r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$\Rightarrow \quad r^2(\cos^2\theta + \sin^2\theta) = a^2$

$\Rightarrow \quad r^2 = a^2$

$\Rightarrow \quad r = a$

The region of integration is the first quadrant of a circle $x^2 + y^2 = a^2$ represented by the region OPQ, as shown in figure,



Figure

$\therefore$ The limits of θ are $\theta = 0$ to $\theta = \dfrac{\pi}{2}$ and the limits of r are $r = 0$ to $r = a$.

$\therefore$ Equation (1) becomes,

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dxdy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2.rdrd\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^3.drd\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} [r^4]_0^a d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} [a^4 - 0]d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} d\theta = \frac{a^4}{4}[\theta]_0^{\pi/2}$$

$$= \frac{a^4}{4}\left[\frac{\pi}{2} - 0\right] = \frac{\pi a^4}{8}$$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dxdy = \frac{\pi a^4}{8}.$$

Q17. Evaluate $\int_0^1 \int_0^1 \int_0^y$ xyz dxdydz

Answer :

Given integral is,

$$\int_0^1 \int_0^1 \int_0^y xyz\;dxdy\;dz$$

The above integral can be evaluated as,

$$\int_0^1 \int_0^1 \int_0^y xyz\;dx\;dy\;dz = \int_0^1 \int_0^1 xy\left[\frac{z^2}{2}\right]_0^y dx\;dy$$

$$= \int_0^1 \int_0^1 xy\left[\frac{y^2}{2} - 0\right] dx\;dy$$

$$= \frac{1}{2}\int_0^1 \int_0^1 xy^3\;dx\;dy$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^4}{4}\right]_0^1 dx$$

$$= \frac{1}{8}\int_0^1 xdx$$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{1}{2} - 0\right]$$

$$= \frac{1}{16}$$

$$\therefore \quad \int_0^1 \int_0^1 \int_0^y xyz\;dx\;dy\;dz = \frac{1}{16}.$$

Q18. Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x}$ x dz dx dy

Answer :

Given integral is,

$$\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy$$

The above integral can be evaluated as,

$$\int_0^1\int_{y^2}^1\int_0^{1-x} xdzdxdy = \int_0^1\int_{y^2}^1\left[\int_0^{1-x} x\,dz\right]dx\,dy$$

$$= \int_0^1\int_{y^2}^1 x(z)_0^{1-x}dx\,dy$$

$$= \int_0^1\int_{y^2}^1 x(1-x-0)\,dx\,dy$$

$$= \int_0^1\int_{y^2}^1 x(1-x)\,dx\,dy$$

$$= \int_0^1\int_{y^2}^1 (x-x^2)\,dx\,dy$$

$$= \int_0^1\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_{y^2}^1 dy$$

$$= \int_0^1\left(\left(\frac{(1)^2}{2}-\frac{(1)^3}{3}\right)-\left(\frac{(y^2)^2}{2}-\frac{(y^2)^3}{3}\right)\right)dy$$

$$= \int_0^1\left(\frac{1}{2}-\frac{1}{3}-\frac{y^4}{2}+\frac{y^6}{3}\right)dy$$

$$= \int_0^1\left(\frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\right)dy$$

$$= \left(\frac{y}{6}-\frac{1}{2}\cdot\frac{y^5}{5}+\frac{1}{3}\cdot\frac{y^7}{7}\right)_0^1$$

$$= \left(\frac{y}{6}-\frac{y^5}{10}+\frac{y^7}{21}\right)_0^1$$

$$= \left(\frac{1}{6}-\frac{(1)^5}{10}+\frac{(1)^7}{21}\right)-\left(\frac{0}{6}-\frac{(0)^5}{10}+\frac{(0)^7}{21}\right)$$

$$= \left(\frac{1}{6}-\frac{1}{10}+\frac{1}{21}\right)-0+0-0 = \frac{4}{35}$$

$$\therefore \int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy = \frac{4}{35}$$

Q19. Evaluate $\int_0^1 dx\int_1^3 dy\int_1^3 xyz\,dz$.

Answer :

Given integral is,

$$\int_0^1 dx\int_1^3 dy\int_1^3 xyz\,dz$$

The above integral can be evaluated as,

$$\int_0^1 dx\int_1^3 dy\int_1^3 xyz\,dz = \int_0^1 dx\int_1^2 dy\int_1^3 xyz\,dz$$

$$= \int_0^1 dx\int_1^2 [xy]\left[\frac{z^2}{2}\right]_1^3 dy$$

$$= \int_0^1 dx\int_1^2 xy\left[\frac{9-1}{2}\right]dy$$

$$= 4\int_0^1 xdx\int_1^2 y\,dy$$

$$= 4\int_0^1 x\left[\frac{y^2}{2}\right]_1^2 dx$$

$$= 4\int_0^1 x\left[\frac{4-1}{2}\right]dx$$

$$= 6\int_0^1 xdx$$

$$= 6\left[\frac{x^2}{2}\right]_0^1$$

$$= 6\left[\frac{1}{2}-0\right] = 3$$

$$\therefore \int_0^1 dx\int_1^2 dy\int_1^3 xyz\,dz = 3.$$

Q20. Evaluate the triple integral $\int_1^3\int_2^3\int_1^2 x^2 yz\,dx\,dy\,dz$

Answer :

Given integral is,

$$\int_1^3\int_2^3\int_1^2 x^2 yz\,dx\,dy\,dz$$

The above integral can be evaluated as,

$$\Rightarrow \int_1^3\int_2^3\int_1^2 x^2 yz\,dx\,dy\,dz = \int_1^3\int_2^3 x^2y\left[\frac{z^2}{2}\right]_1^2 dx\,dy$$

$$= \int_1^3\int_2^3 x^2y\left[\frac{4-1}{2}\right]dx\,dy$$

$$= \frac{3}{2}\int_1^3 x^2\left[\frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{3}{2}\int_1^3 x^2\frac{(9-4)}{2}\,dx$$

$$= \frac{15}{4}\int_1^3 x^2 dx$$

$$= \frac{15}{4}\left[\frac{x^3}{3}\right]_1^3$$

$$= \frac{15}{4}\left[\frac{27-1}{3}\right] = \frac{65}{2}$$

$$\therefore \quad \int_1^3\int_2^3\int_1^2 x^2\, yz\, dx\, dy\, dz = \frac{65}{2}$$

Q21. Evaluate $\int_0^1\int_0^2\int_0^3$ xyz dx dy dz.

Answer :

Given integral is,

$$\int_0^1\int_0^2\int_0^3 xyz\, dx\, dy\, dz$$

The above integral can be evaluated as,

$$\int_0^1\int_0^2\int_0^3 xyz\, dx\, dy\, dz = \int_0^1\int_0^2 yz\left[\int_0^3 x\, dx\right]dy\, dz$$

$$= \int_0^1\int_0^2 yz\left[\frac{x^2}{2}\right]_0^3 dy\, dz$$

$$= \int_0^1\int_0^2 yz\left[\frac{9}{2}-0\right]dy\, dz$$

$$= \int_0^1\int_0^2 \frac{9}{2}\, yz\, dy\, dz$$

$$= \frac{9}{2}\int_0^1 z\left[\int_0^2 y\, dy\right]dz$$

$$= \frac{9}{2}\int_0^1 z\left[\left[\frac{y^2}{2}\right]_0^2\right]dz$$

$$= \frac{9}{2}\int_0^1 z\left[\frac{4}{2}-0\right]dz$$

$$= \frac{9}{2}\left(\frac{4}{2}\right)\int_0^1 z\, dz$$

$$= 9\left[\frac{z^2}{2}\right]_0^1$$

$$= 9\left[\frac{1}{2}-0\right] = \frac{9}{2}$$

$$\therefore \quad \int_0^1\int_0^2\int_0^3 xyz\, dx\, dy\, dz = \frac{9}{2}.$$

Q22. Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}$ dx dy dz.

Answer :

Given integral is,

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\, dx\, dy\, dz$$

The above integral can be evaluated as,

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\, dx\, dy\, dz = \int_0^1\int_0^1\int_0^1 e^x e^y e^z\, dx\, dy\, dz$$

$$= \int_0^1 e^x dx \int_0^1 e^y dy \int_0^1 e^z dz$$

$$= \int_0^1 e^x dx \int_0^1 e^y [e^z]_0^1 dy$$

$$= \int_0^1 e^x dx \int_0^1 e^y [e^1 - e^0] dy$$

$$= \int_0^1 e^x dx \int_0^1 e^y [e-1] dy$$

$$= (e-1)\int_0^1 e^x dx\left[\int_0^1 e^y dy\right]$$

$$= (e-1)\int_0^1 e^x [e^y]_0^1\, dx$$

$$= (e-1)\int_0^1 e^x [e^1 - e^0] dx$$

$$= (e-1)\int_0^1 e^x (e-1) dx$$

$$= (e-1)^2\int_0^1 e^x dx$$

$$= (e-1)^2 [e^x]_0^1$$

$$= -(e-1)^2 (e^1 - e^0)$$

$$= (e-1)^2\,(e-1)$$

$$= (e-1)^3$$

$$\therefore \quad \int_0^1\int_0^1\int_0^1 e^{x+y+z}\, dx\, dy\, dz = (e-1)^3.]$$

Q23. Evaluate $\int_0^2 \int_0^y \int_0^x dx\, dy\, dz$.

Answer :

Given integral is,

$$\int_0^2 \int_0^y \int_0^x dx\, dy\, dz$$

The above integral can be evaluated as,

$$\Rightarrow \int_0^2 \int_0^y \int_0^x dx\, dy\, dz = \int_0^2 \int_0^y [z]_0^x\, dxdy$$

$$= \int_0^2 \int_0^y x\, dxdy$$

$$= \int_0^2 \left[\frac{x^2}{2}\right]_0^y dy$$

$$= \frac{1}{2}\int_0^2 y^2\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^2$$

$$= \frac{1}{2}\,\frac{8}{3}$$

$$= \frac{4}{3}$$

$$\therefore \int_0^2 \int_0^y \int_0^x dx\, dy\, dz = \frac{4}{3}.$$

Q24. Evaluate $\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^{y} xyz\, dz\, dy\, dx$

Answer :

Given integral is,

$$\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^{y} xyz\, dz\, dy\, dx$$

The above integral can be evaluated as,

$$\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^{y} xyz\, dz\, dy\, dx = \int_0^1 \int_0^1 \left[\int_{\sqrt{x^2+y^2}}^{y} z.dz\right] xy\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left[\frac{z^2}{2}\right]_{\sqrt{x^2+y^2}}^{y} xy\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left[\frac{y^2}{2} - \left(\frac{x^2+y^2}{2}\right)\right] xy\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left(\frac{y^2}{2} - \frac{x^2}{2} - \frac{y^2}{2}\right) xy\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left(\frac{-x^2}{2}\right) xy\, dy\, dx$$

$$= \frac{-1}{2}\int_0^1 \int_0^1 x^3 y\, dy\, dx$$

$$= \frac{-1}{2}\int_0^1 \left[\int_0^1 x^3\, dx\right] y\, dy$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{x^4}{4}\right]_0^1 y\, dy$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{1}{4} - \frac{0}{4}\right] y\, dy$$

$$= \frac{-1}{2}\int_0^1 \frac{y}{4} dy$$

$$= \frac{-1}{2\times 4}\int_0^1 y\, dy$$

$$= \frac{-1}{8}\left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{-1}{8}\left[\frac{1}{2} - \frac{0}{2}\right]$$

$$= \frac{-1}{8\times 2}$$

$$= \frac{-1}{16}$$

$$\therefore \int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^{y} xyz\, dz\, dy\, dx = \frac{-1}{16}.$$

PART-B
ESSAY QUESTIONS WITH SOLUTIONS

4.1 DOUBLE INTEGRALS

Q25. Write short notes on 'Double Integrals'.

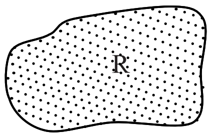
Answer :

Double Integrals

The generalized integral of a definite integral to two dimensions is known as double integral. In this case, the definite integral of a single variable function is extended to a function of two variables.

Explanation

Figure below illustrates a region 'R' in a plane (say xy plane) which is bounded by one or more curves.



Figure

Let,

- $f(x, y)$ – Function which is defined at each and every point of 'R'.
- $\delta R_1, \delta R_2, ..., \delta R_n$ – Pairwise non-overlapping subregions of R.
- (x_i, y_i) – Arbitrary point within a subregion δR_i.

Let the sum,

$$f(x_1, y_1)\delta R_1 + f(x_2, y_2)\delta R_2 + ... + f(x_n, y_n)\delta R_n \quad ... (1)$$

As $n \to \infty$, the sum (equation (1)) tends to a finite limit such that max $(\delta R_i) \to 0$ regardless of the arbitrary choice of (x_i, y_i). This finite limit is termed as 'Double integral' of the function $f(x, y)$ over the region R. It is denoted as $\iint_R f(x, y)dR$ or $\iint_R f(x, y)dx\, dy$

Properties

(i) $\iint_R (f+g)dx\, dy = \iint_R f\ dx\, dy + \iint_R g\ dx\, dy$

(ii) $\iint_R k\ f\ dx\, dy = k\iint_R f\ dx\, dy$, where k is a constant

(iii) $\iint_R f\ dx\, dy = \iint_{R_1} f\ dx\, dy + \iint_{R_2} f\ dx\, dy$

Where,

R_1, R_2 - Two distinct regions of R and $R_1 \cup R_2 = R$.

(iv) Mean value theorem for double integral

The region 'R' consists of atleast one point (x_0, y_0) such that,

$$\iint_{R_2} f(x, y)dx\, dy = f(x_0, y_0)A \quad \text{(for continuous '}f\text{' in } R)$$

Where,

A – Area.

Q26. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dxdy$.

Answer :

Given integral is,

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dxdy$$

The above integral can be evaluated as,

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dxdy = \int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} (x^2+y^2)\,dxdy$$

$$= \int_{x=0}^{x=1} \left[x^2[y]_x^{\sqrt{x}} + \left[\frac{y^3}{3}\right]_x^{\sqrt{x}} \right] dx$$

$$= \int_{x=0}^{x=1} \left[x^2(\sqrt{x}-x) + \frac{1}{3}\left(\left(\sqrt{x}\right)^3 - x^3\right) \right] dx$$

$$= \int_{x=0}^{x=1} \left[x^2\sqrt{x} - x^2 x + \frac{\left(\sqrt{x}\right)^3}{3} - \frac{x^3}{3} \right] dx$$

$$= \int_{x=0}^{x=1} \left[x^{\frac{5}{2}} - x^3 + \frac{x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right] dx$$

$$= \int_{x=0}^{x=1} \left[x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - \frac{4x^3}{3} \right] dx$$

$$= \left[\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{1}{3}\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right) - \frac{4}{3}\left(\frac{x^{3+1}}{3+1}\right) \right]_0^1$$

$$= \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{1}{3}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) - \frac{4}{3}\left(\frac{x^4}{4}\right) \right]_0^1 = \left[\frac{1^{\frac{7}{2}}}{\frac{7}{2}} - 0 + \frac{1}{3}\left(\frac{1^{\frac{5}{2}}}{\frac{5}{2}} - 0\right) - \frac{4}{3}\left(\frac{1^4}{4} - 0\right) \right]$$

$$= \frac{2}{7} + \frac{1}{3}.\frac{2}{5} - \frac{4}{3}.\frac{1}{4} = \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{30+14-35}{105} = \frac{9}{105} = \frac{3}{35}$$

$$\therefore \quad \int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dxdy = \frac{3}{35}$$

Q27. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

Answer :

Given integral is,

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

$$\Rightarrow \quad I = \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\right] dy\, dx$$

$$= \int_0^a \left[\frac{y}{2}\sqrt{(a^2-x^2)-y^2} + \frac{(a^2-x^2)}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \int_0^a \frac{\sqrt{(a^2-x^2)}}{2}\sqrt{a^2-x^2-\left(\sqrt{a^2-x^2}\right)^2} + \frac{(a^2-x^2)}{2}\sin^{-1}\left[\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} - 0\right] dx$$

$$= \int_0^a \left[\frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-x^2-a^2+x^2} + \frac{(a^2-x^2)}{2}\sin^{-1}(1)\right] dx$$

$$= \int_0^a \left[0 + \frac{a^2-x^2}{2}\left(\frac{\pi}{2}\right)\right] dx$$

$$= \frac{\pi}{4}\int_0^a (a^2-x^2)\, dx$$

$$= \frac{\pi}{4}\left[a^2\int_0^a dx - \int_0^a x^2\, dx\right]$$

$$= \frac{\pi}{4}\left[a^2[x]_0^a - \left[\frac{x^3}{3}\right]_0^a\right]$$

$$= \frac{\pi}{4}\left[a^2[a] - \frac{a^3}{3}\right]$$

$$= \frac{2a^3}{12}\pi$$

$$= \frac{\pi a^3}{6}$$

$$\therefore \quad \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx = \frac{\pi a^3}{6}$$

Q28. Evaluate: $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\, d\theta\, dr$ **.**

Answer :

Given integral is,

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\, d\theta\, dr$$

The above integral can be evaluated as,

$$\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r d\theta\, dr = \int_0^{\frac{\pi}{2}}\left[\int_0^{\sin\theta} r dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{\sin^2\theta}{2} - \frac{0}{2}\right]d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left\{\left[\frac{\pi}{2} - \frac{\sin 2\frac{\pi}{2}}{2}\right] - \left[0 - \frac{\sin 0}{2}\right]\right\}$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \frac{0}{2}\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{8}$$

$$\therefore \quad \int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r d\theta\, dr = \frac{\pi}{8}.$$

Q29. Find the values $\int\int$ xy dx dy taken over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Answer :

Given integral is,

$$\int\int xy\, dx\, dy$$

Ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\Rightarrow \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$\therefore$ y varies from 0 to $\frac{b}{a}\sqrt{a^2 - x^2}$

Since the region of integration is in positive quadrant of the ellipse, x varies from 0 to a

$$\therefore \quad \int\int_R xy\, dx\, dy = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (xy)\, dy\, dx$$

$$= \int_0^a\left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\, dy\right]x\, dx$$

$$= \int_0^a\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} x\, dx$$

$$= \frac{1}{2}\int_0^a\left[\frac{b^2}{a^2}(a^2 - x^2) - 0\right]x\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a (xa^2 - x^3)\,dx$$

$$= \frac{b^2}{2a^2}\left[a^2\int_0^a x^2 dx - \int_0^a x^3 dx\right]$$

$$= \frac{b^2}{2a^2}\left[a^2\left[\frac{x^2}{2}\right]_0^a - \left[\frac{x^4}{4}\right]_0^a\right]$$

$$= \frac{b^2}{2a^2}\left[a^2\left[\frac{a^2}{2} - 0\right] - \left[\frac{a^4}{4} - 0\right]\right]$$

$$= \frac{b^2}{2a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{b^2}{2a^2}\left[\frac{a^4}{4}\right]$$

$$= \frac{a^2 b^2}{8}$$

$$\therefore \quad \int\int xy\, dx\, dy = \frac{a^2 b^2}{8}$$

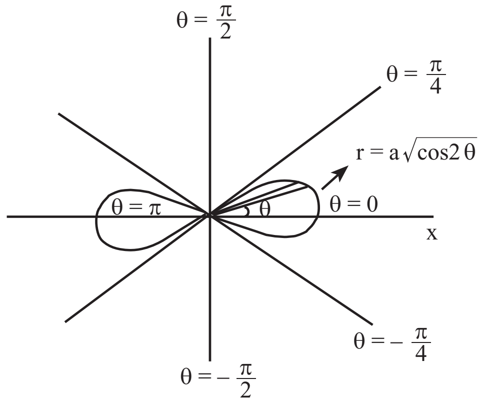
Q30. Find the area of $r^2 = a^2\cos^2\theta$ by double integration.

Answer :

Given curve is,

$r^2 = a^2\cos^2\theta$ ... (1)

Figure below represents the shape of the curve.



Figure

The curve is symmetrical with respect to *X*-axis.

From equation (1),

$$r = \sqrt{a^2 \cos 2\theta}$$
$$= a\sqrt{\cos 2\theta}$$

$\therefore$ r varies from 0 to $a\sqrt{\cos 2\theta}$

θ varies from 0 to $\frac{\pi}{4}$

Area of the curve is given as,

Area = 4 × Area of upper half of one loop ... (2)

Area of upper half of one loop is given as,

$$A = \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} r dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4}\left[\sin\frac{\pi}{2} - 0\right]$$

$$= \frac{a^2}{4}$$

$$\therefore \quad A = \frac{a^2}{4} \quad \text{... (3)}$$

Substituting equation (3) in equation (2),

$$\text{Area} = 4.\ \frac{a^2}{4}$$
$$= a^2$$

$\therefore$ Area of the curve = a^2 square units.

Q31. Evaluate $\int_0^{\pi/4}\int_0^{\sin\theta} \frac{r\, dr d\theta}{\sqrt{a^2 - r^2}}$

Answer :

Given integral is,

$$\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}}$$

The above integral can be evaluated as,

$$\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\, dr\, d\theta}{\sqrt{a^2 - r^2}} = \int_0^{\pi/4}\left[\frac{1}{2}\int_0^{a\sin\theta} \frac{2r}{\sqrt{a^2 - r^2}} dr\right] d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\left[-2\sqrt{a^2 - r^2}\right]_0^{a\sin\theta} d\theta$$

$$\left[\because \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c\right]$$

$$= \int_0^{\pi/4}\left[-\sqrt{a^2 - a^2\sin^2\theta} + \sqrt{a^2 - 0^2}\right] d\theta$$

$$= \int_0^{\pi/4}\left(-a\sqrt{1 - \sin^2\theta} + a\right) d\theta$$

$$= \int_0^{\pi/4}\left(-a\sqrt{\cos^2\theta} + a\right) d\theta$$

$$= \int_0^{\pi/4}(a - a\cos\theta) d\theta$$

$$= [a\theta - a\sin\theta]_0^{\pi/4}$$

$$= \left[a\left(\frac{\pi}{4}\right) - a\sin\left(\frac{\pi}{4}\right) - 0 - 0\right]$$

$$= \frac{a\pi}{4} - \frac{a}{\sqrt{2}} = a\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right]$$

$$\therefore \int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\, dr\, d\theta}{\sqrt{a^2 - r^2}} = a\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right]$$

Q32. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta dr\, d\theta$

Answer :

Given integral is,

$$\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta.dr.d\theta$$

$$\text{Let } I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta\, dr d\theta$$

$$= \int_0^{\pi} \cos\theta d\theta \int_0^{a(1+\cos\theta)} r^2 dr$$

$$= \int_0^{\pi} \cos\theta d\theta \left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)}$$

$$= \frac{1}{3}\int_0^{\pi} \cos\theta d\theta[(a(1+\cos\theta))^3 - 0]$$

$$= \frac{1}{3}\int_0^{\pi} \cos\theta d\theta a^3(1+\cos\theta)^3$$

$$I = \frac{a^3}{3}\int_0^{\pi} \cos\theta(1+\cos\theta)^3 d\theta \quad \text{... (1)}$$

Since, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Equation (1) becomes,

$$I = \frac{a^3}{3}\int_0^{\pi}\cos(\pi-\theta)[1+\cos(\pi-\theta)]^3 d\theta$$

$$I = \frac{a^3}{3}\int_0^{\pi}-\cos\theta\,(1-\cos\theta)^3 d\theta \quad \text{... (2)}$$

Adding equations (1) and (2),

$$I+I = \frac{a^3}{3}\int_0^{\pi}\cos\theta(1+\cos\theta)^3 d\theta + \frac{a^3}{3}\int_0^{\pi}-\cos\theta(1-\cos\theta)^3 d\theta$$

$$2I = \frac{a^3}{3}\int_0^{\pi}[\cos\theta(1+\cos\theta)^3 - \cos\theta(1-\cos\theta)^3]d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi}\cos\theta\,[(1+\cos\theta)^3-(1-\cos\theta)^3]d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi}\cos\theta\,[2\cos\theta\{3(1)^2+\cos^2\theta\}]d\theta$$

$[\because (a+b)^3-(a-b)^3 = 2b(3a^2+b^2)]$

$$= \frac{2a^3}{3}\int_0^{\pi}\cos\theta[\cos\theta\{3+\cos^2\theta\}]d\theta$$

$$\therefore \quad 2I = \frac{2a^3}{3}\int_0^{\pi}(3\cos^2\theta+\cos^4\theta)d\theta \quad \text{... (3)}$$

$$= \frac{2a^3}{3}.2\int_0^{\pi/2}(3\cos^2\theta+\cos^4\theta)d\theta$$

$$\left(\because \int_0^{2a}f(x)dx = 2\int_0^{a}f(x)dx\right)$$

$$= \frac{4a^3}{3}\left[\int_0^{\pi/2}3\cos^2\theta d\theta + \int_0^{\pi/2}\cos^4\theta d\theta\right]$$

$$\left[\because \int_0^{\pi/2}\cos^n x\,dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)...\frac{1}{2}\frac{\pi}{2}, \text{ if } n \text{ is even}\right]$$

$$= \frac{4a^3}{3}\left[3.\left(\frac{2-1}{2}\right)\frac{\pi}{2}+\left(\frac{4-1}{4}\right)\left(\frac{4-3}{4-2}\right)\frac{\pi}{2}\right]$$

$$= \frac{4a^3}{3}\left[3.\frac{\pi}{4}+\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right] = \frac{4a^3}{3}\left[\frac{3\pi}{4}+\frac{3\pi}{16}\right]$$

$$= \frac{4a^3}{3}\left[\frac{12\pi+3\pi}{16}\right] = \frac{4a^3}{3}\times\frac{15\pi}{16} = \frac{5a^3\pi}{4}$$

$$\therefore \; 2I = \frac{5a^3\pi}{4}$$

$$\Rightarrow \quad I = \frac{5a^3\pi}{8}$$

$$\therefore \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr d\theta = \frac{5a^3\pi}{8}$$

Q33. Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{(a^2-r^2)}{a}}$ **rdzdrd**θ

Answer :

Given integral is,

$$\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} rdz\,dr\,d\theta$$

The above integral can be evaluated as,

$$\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} rdz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} r[z]_0^{\frac{a^2-r^2}{a}} dr.d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} r\left[\frac{a^2-r^2}{a}-0\right]dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\left[\frac{ra^2}{a}-\frac{r^3}{a}\right]dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\left[ra-\frac{r^3}{a}\right]dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[a\left[\frac{r^2}{2}\right]-\frac{1}{a}\left[\frac{r^4}{4}\right]\right]_0^{a\sin\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left\{\frac{a}{2}[a^2\sin^2\theta-0]-\frac{1}{4a}[a^4\sin^4\theta-0]\right\}d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{a^3}{2}\sin^2\theta-\frac{a^4}{4a}\sin^4\theta\right]d\theta$$

$$= \frac{a^3}{2}\int_0^{\pi/2}\sin^2\theta.d\theta - \frac{a^3}{4}\int_0^{\pi/2}\sin^4\theta.d\theta$$

$$= \left[\frac{a^3}{2}.\frac{1}{2}.\frac{\pi}{2}\right]-\left[\frac{a^3}{4}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$\left(\because \int_0^{\pi/2}\sin^n x dx = \frac{(n-1)(n-3)(n-5)....}{n(n-2)(n-4)....}\times\frac{\pi}{2}\left(\text{only if } n \text{ is even}\right)\right)$$

$$= \frac{\pi a^3}{8} - \frac{3a^3\pi}{64} = \frac{5a^3\pi}{64}$$

$$\therefore \int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dz.dr.d\theta = \frac{5\pi a^3}{64}$$

Q34. Using double integral, find the area bounded by y = x and y = x².

Answer :

Given that,

$$y = x \quad \text{... (1)}$$

$$y = x^2 \quad \text{... (2)}$$

Solving equations (1) and (2),

$\Rightarrow \quad x^2 = x$

$\Rightarrow \quad x^2 - x = 0$

$\Rightarrow \quad x(x - 1) = 0$

$\Rightarrow \quad x = 0 , x = 1$

If $x = 0, y = 0$

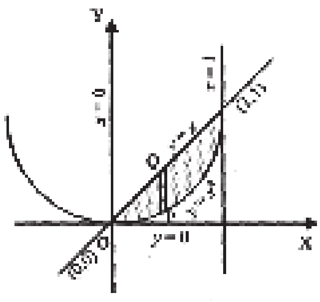
If $x = 1, y = 1$.

$\therefore$ The point of intersection of equations (1) and (2) is (0, 0) and (1, 1).

$\therefore$ x varies from 0 to 1.

y varies from x^2 to x.

Figure below represents the region of integration.



Figure

The area bounded by the lines $y = x$ and $y = x^2$ is,

$$\int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \left[\int_{x^2}^{x} dy\right] dx$$

$$= \int_0^1 [y]_{x^2}^{x}\, dx$$

$$= \int_0^1 [x - x^2]\, dx$$

$$= \int_0^1 x\, dx - \int_0^1 x^2\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1}{2} - 0\right] - \left[\frac{1}{3} - 0\right]$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\therefore \int_0^1 \int_{x^2}^{x} dy\, dx = \frac{1}{6} \text{ square units.}$$

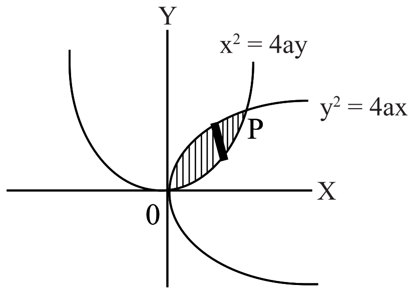
Q35. Using double integral find the area bounded by the parabolas y² = 4ax and x² = 4ay.

Answer :

Given parabolas are,

$$y^2 = 4ax \text{ and } x^2 = 4ay \quad \text{... (1)}$$

Figure below illustrates the area between the two parabolas.



Figure

From figure,

$$y = \frac{x^2}{4a} \text{ and } y = \sqrt{4ax}$$

$\Rightarrow \quad \frac{x^2}{4a} = \sqrt{4ax}$

$\Rightarrow \quad x^2 = 4a.(4a)^{1/2}.x^{1/2}$

$\Rightarrow \quad x^{2-\frac{1}{2}} = (4a)^{3/2} \Rightarrow x^{3/2} = (4a)^{3/2} \Rightarrow x = 4a$

$\therefore$ $x = 0, x = 4a$ is the region.

Required area can be calculated as,

$$A = \int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx\, dy = \int_0^{4a} [y]_{\frac{x^2}{4a}}^{\sqrt{4ax}}\, dx$$

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$

$$= \int_0^{4a} \left[\sqrt{4a}.(x)^{1/2} - \frac{x^2}{4a}\right] dx$$

$$= \left[\sqrt{4a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \left[\sqrt{4a} \cdot \frac{(4a)^{3/2}}{3/2} - \frac{(4a)^3}{12a} \right]$$

$$= (4a)^{1/2} \cdot \frac{2(4a)^{3/2}}{3} - \frac{64.a^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{64a^2}{12} = \frac{128a^2 - 64a^2}{12} = \frac{64}{12}a^2$$

$$= \frac{16}{3}a^2$$

$\therefore$ Area $= \frac{16}{3}a^2$ sq.units.

Q36. Using double integral find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer :

Given equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{... (1)}$$

$$\Rightarrow \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\Rightarrow x = \sqrt{\frac{a^2(b^2 - y^2)}{b^2}}$$

The ellipse has center at origin (0, 0)

$\therefore$ x varies from 0 to $\sqrt{b^2 - y^2}$

Substituting $x = 0$ in equation (1),

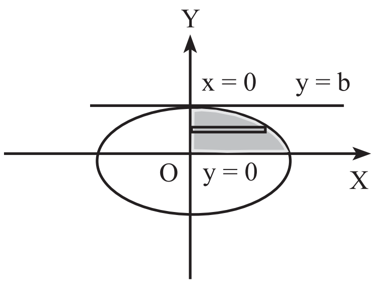
$$\frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = b^2$$

$$\Rightarrow y = b$$

$\therefore$ y varies from 0 to b.

Figure represents the region of integration which is a positive quadrant.



Figure

Area of ellipse is given as,

Area = 4 × Area of the quadrant ... (2)

Area of the quadrant is given as,

$$A = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx\, dy$$

$$= \int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \int_0^b \left[\frac{a}{b}\sqrt{b^2 - y^2} - 0\right] dy$$

$$= \frac{a}{b}\int_0^b \sqrt{b^2 - y^2}\, dy$$

$$= \frac{a}{b}\left[\frac{b^2}{2}\sin^{-1}\frac{y}{b} + \frac{y}{2}\sqrt{b^2 - y^2}\right]_0^b$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a}{2}\sqrt{a^2 - x^2}\right]$$

$$= \frac{a}{b}\left[\frac{b^2}{2}\sin^{-1}\frac{b}{b} + \frac{b}{2}\sqrt{b^2 - b^2} - 0\right]$$

$$= \frac{a}{b}\,\frac{b^2}{2} \cdot \frac{\pi}{2} = \frac{\pi ab}{4}$$

$$\therefore A = \frac{\pi ab}{4} \quad \text{... (3)}$$

Substituting equation (3) in equation (2)

$$\text{Area} = 4 \times \frac{\pi ab}{4} = \pi ab$$

$\therefore$ Area of ellipse is πab square units.

Q37. Evaluate $\iint$ xy dxdy over the first quadrant of the circle $x^2 + y^2 = 4$.

Answer :

Given integral is,

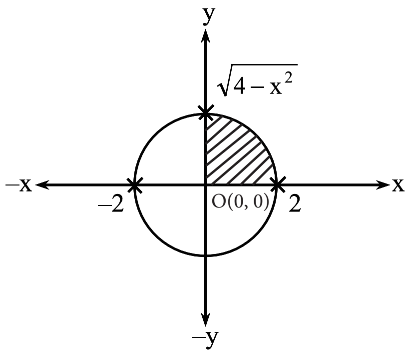
$$\iint xy\, dxdy$$

Equation of circle, $x^2 + y^2 = 4$

$$\Rightarrow y^2 = 4 - x^2$$

$$\Rightarrow y = \sqrt{4 - x^2}$$

Figure represents the region of integration in a circle.



Figure

Let, $y = 0$,

Then,

$$4 - x^2 = 0$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = \sqrt{4} = 2$$

$\therefore$ y varies from 0 to $\sqrt{4-x^2}$ and x varies from 0 to 2.

$$\therefore \iint xy\ dx\ dy = \int_0^2 \int_0^{\sqrt{4-x^2}} x\ y\ dx\ dy$$

$$= \int_0^2 x\left[\int_0^{\sqrt{4-x^2}} y\ dy\right]dx$$

$$= \int_0^2 x \cdot \left[\frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \frac{1}{2}\int_0^2 x\left[\left(\sqrt{4-x^2}\right)^2 - 0\right] dx$$

$$= \frac{1}{2}\int_0^2 (4x - x^3)\ dx$$

$$= \frac{1}{2}\left[4\int_0^2 xdx - \int_0^2 x^3 dx\right]$$

$$= \frac{1}{2}\left[4\cdot\left[\frac{x^2}{2}\right]_0^2 - \left[\frac{x^4}{4}\right]_0^2\right]$$

$$= \frac{1}{2}\left[4\cdot\left[\frac{4}{2}\right] - \left[\frac{16}{4}\right]\right]$$

$$= \frac{1}{2}[8-4]$$

$$= 2$$

$$\therefore \iint xy\,dxdy = 2$$

4.2 CHANGE OF ORDER OF INTEGRATION

Q38. Evaluate $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2\,dydx$ by changing the order of integration.

Answer :

Given integral is,

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2\,dydx$$

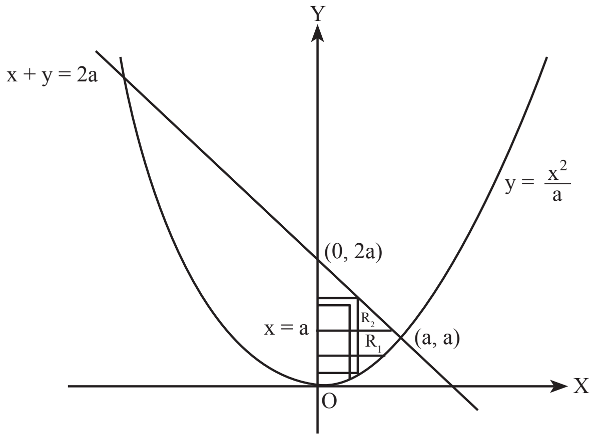


Figure (1)

The above figure (i) illustrates the curve for x varying from 0 to a and y varying from $\frac{x^2}{a}$ to $2a - x$.

Change of Order of Integration

Order of integration is changed by drawing horizontal strip to cover the region. The region is split into two segments R_1 and R_2 as illustrated in Figure.

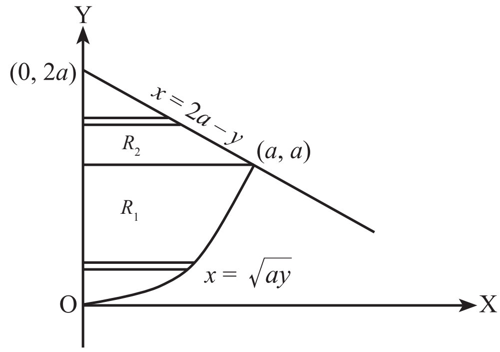


Figure (2)

Integral Over R_1

The integration limits are $y = 0$, $y = a$, $x = 0$ and,

$$x = \sqrt{ay} \quad \left(\because y = \frac{x^2}{a} \Rightarrow x^2 = ay\right)$$

$$R_1 = \int_0^a \int_0^{\sqrt{ay}} xy^2\,dx\,dy$$

Integral Over R_2

The integration limits are $y = a$, $y = 2a$, $x = 0$ and $x = 2a - y$ $(\because y = 2a - x \Rightarrow x = 2a - y)$

$$R_2 = \int_a^{2a} \int_0^{2a-y} xy^2\,dx\,dy$$

Then,

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2\,dy\,dx = \int_0^a \int_0^{\sqrt{ay}} xy^2\,dx\,dy + \int_0^{2a}\int_0^{2a-y} xy^2\,dx\,dy$$

$$= \int_0^a y^2\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y^2\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \int_0^a \frac{y^2}{2}\left[(\sqrt{ay})^2 - 0\right]dy + \int_a^{2a} \frac{y^2}{2}\left[(2a-y)^2 - 0\right]dy$$

$$= \int_0^a \frac{y^2}{2}.ay\,dy + \int_a^{2a} \frac{y^2}{2}\left[4a^2 + y^2 - 4ay\right]dy$$

$$= \int_0^a \frac{a}{2}y^3\,dy + \frac{1}{2}\int_a^{2a}\left[4a^2y^2 + y^4 - 4ay^3\right]dy$$

$$= \frac{a}{2}\left[\frac{y^4}{4}\right]_0^a + \frac{1}{2}\left[\frac{4a^2y^3}{3} + \frac{y^5}{5} - \frac{4ay^4}{4}\right]_a^{2a}$$

$$= \frac{a}{2}\left[\frac{a^4}{4} - 0\right] + \frac{1}{2}\left[\left[\frac{4a^2(2a)^3}{3} + \frac{(2a)^5}{5} - \frac{a(2a)^4}{4}\right]\right] - \left[\frac{4a^2a^3}{3} + \frac{a^5}{5} - \frac{4a^5}{4}\right]$$

$$= \frac{a^5}{8} - 0 + \frac{1}{2}\left[\frac{32a^5}{3} + \frac{32a^5}{5} - 16a^5 - \frac{4a^5}{3} - \frac{a^5}{5} + a^5\right]$$

$$= \frac{a^5}{8} + \frac{1}{2}\left[\frac{28a^5}{3} + \frac{31a^5}{5} - 15a^5\right]$$

$$= \frac{a^5}{8} + \frac{1}{2}\left[\frac{8a^5}{15}\right]$$

$$= \frac{a^5}{8} + \frac{4a^5}{15} = \frac{47a^5}{120}$$

$$\therefore \quad \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2\,dy\,dx = \frac{47a^5}{120}.$$

Q39. Change the order of integration for the given integral $\int_0^a \int_0^{2\sqrt{ax}} (x^2)\,dy\,dx$ and evaluate it.

Answer :

Given integral is,

$$\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx$$

The region of integration is bounded by

$x = 0$, $x = a$ and $y = 0$, $y = 2\sqrt{ax}$

Since $y = 2\sqrt{ax}$

$\Rightarrow \quad y^2 = 4ax$

$\Rightarrow \quad x = \dfrac{y^2}{4a}$

If x = a

$\Rightarrow \quad a = \dfrac{y^2}{4a}$

$\Rightarrow \quad 4a^2 = y^2$

$y = \pm 2a$

$\therefore \quad x$ varies from a to $\dfrac{y^2}{4a}$

y varies from 0 to $2a$

$$\therefore \quad \int_0^a \int_0^{2\sqrt{ax}} x^2 dy\, dx = \int_0^{2a} \int_a^{\frac{y^2}{4a}} x^2 dx\, dy$$

$$= \int_0^{2a} \left[\frac{x^3}{3}\right]_a^{\frac{y^2}{4a}} dy$$

$$= \frac{1}{3}\int_0^{2a} \left[\frac{y^6}{(4a)^3} - a^3\right] dy$$

$$= \frac{1}{3}\left[\frac{y^7}{7(64a^3)} - a^3 y\right]_0^{2a}$$

$$= \frac{1}{3}\left[\frac{128\,a^7}{7(64)a^3} - a^3(2a) - 0\right]$$

$$= \frac{1}{3}\left[\frac{2}{7}a^4 - 2a^4\right]$$

$$= \frac{1}{3}\left[\frac{-12}{7}\,a^4\right]$$

$$= \frac{-4}{7}a^4$$

$$\therefore \quad \int_0^a \int_0^{2\sqrt{ax}} x^2 dydx = \frac{-4}{7}a^4$$

Q40. Evaluate $\int_0^1 \int_{\sqrt{y}}^{2-y} xydxdy$ by changing the order of integration.

Answer :

Given integral is,

$$\int_0^1 \int_{\sqrt{y}}^{2-y} xy\, dx\, dy$$

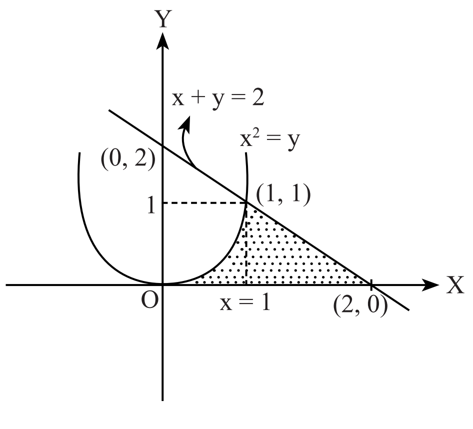


Figure (1)

Figure (i) illustrates the curve for x varying from $\sqrt{y}$ to $2 - y$ and y varying from 0 to 1.

Change of Order of Integration

Order of integration is changed by drawing vertical strip to cover the region. The region is split into two segments R_1 and R_2 as illustrated in figure (i).

Integral Over R_1

The integration limits are $x = 0$, $x = 1$, $y = 0$ and $y = x^2$

$$[\because x = \sqrt{y} \Rightarrow x^2 = y]$$

$$R_1 = \int_0^1 \int_0^{x^2} xydxdy$$

Integral Over R_2

The integration limits are,

$x = 1$, $x = 2$, $y = 0$ and $y = 2 - x$

$$[\because x = 2 - y \Rightarrow y = 2 - x]$$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} xy\, dy\, dx + \int_{x=1}^{2} \int_{y=0}^{2-x} xy\, dy\, dx$$

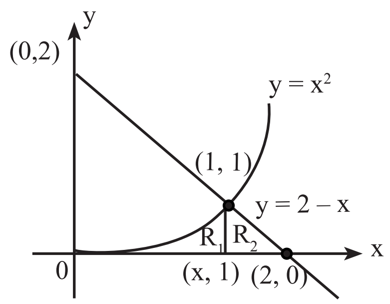


Figure (2)

$$= \int_{x=0}^{1} x\left\{\int_{y=0}^{x^2} y\, dy\right\}dx + \int_{x=1}^{2} x\left\{\int_{y=0}^{2-x} y\, dy\right\}dx$$

$$= \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_0^{x^2} dx + \int_{x=1}^{2} x\left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} x\left[(x^2)^2 - 0\right]dx + \frac{1}{2}\int_{x=1}^{2} x\left[(2-x)^2 - 0\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{1} x^5 dx + \frac{1}{2}\int_{x=1}^{2} x\left[4 + x^2 - 4x\right]dx$$

$$= \frac{1}{2}\left[\int_{x=0}^{1} x^5 dx + \int_{x=1}^{2} (4x + x^3 - 4x^2)\,dx\right]$$

$$= \frac{1}{2}\left[\int_0^1 x^5 dx + 4\int_1^2 x\, dx + \int_1^2 x^3\, dx - 4\int_1^2 x^2 dx\right]$$

$$= \frac{1}{2}\left[\left[\frac{x^6}{6}\right]_0^1 + 4\left[\frac{x^2}{2}\right]_1^2 + \left[\frac{x^4}{4}\right]_1^2 - 4\left[\frac{x^3}{3}\right]_1^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{6}\left(1^6 - 0\right) + \frac{4}{2}\left((2)^2 - (1)^2\right) + \frac{1}{4}\left((2)^4 - (1)^4\right) - \frac{4}{3}\left((2)^3 - (1)^3\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{6}(1) + 2(3) + \frac{1}{4}(15) - \frac{4}{3}(7)\right]$$

$$= \frac{1}{2}\left[\frac{1}{6} + 6 + \frac{15}{4} - \frac{28}{3}\right]$$

$$= \frac{1}{2}\left[\frac{2 + 72 + 45 - 112}{12}\right]$$

$$= \frac{1}{24} = \frac{7}{24}$$

$$\therefore \int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy = \frac{7}{24}.$$

Q41. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dx\,dy$ by changing the order of integration.

Answer :

Given integral is,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dx\,dy \qquad \ldots (1)$$

Equation (1) can be written in standard form as,

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx$$

The limits of x are 0 and 1. The area of integration lies between $y = 0$ and $y = \sqrt{1-x^2}$ i.e., $x^2 + y^2 = 1$ (It represents a circle). Figure (i) illustrates a circle where the region of integration OAB, is divided into vertical strips.

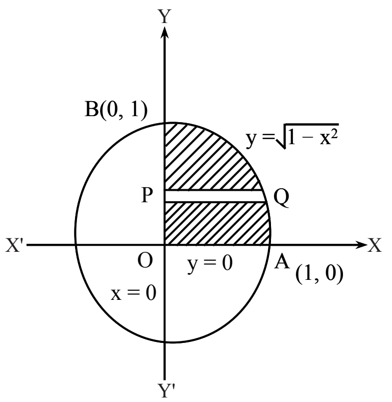


Figure (1)

Change of Order

To change the order of integration the region of integration is divided into horizontal strips.

$\therefore$ Integration limits are,

$$x = 0 \text{ to } \sqrt{1-y^2}$$

$$y = 0 \text{ to } 1$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{y=0}^{1} y^2\,dy \int_{x=0}^{\sqrt{1-y^2}} dx$$

$$= \int_{y=0}^{1} y^2\,[x]_{x=0}^{\sqrt{1-y^2}}\,dy$$

$$= \int_{y=0}^{1} y^2\sqrt{1-y^2}\,dy$$

Let, $y = \sin\theta$

$\Rightarrow$ $dy = \cos\theta\,d\theta$

Limits

$y = 0,\ \sin\theta = 0 \Rightarrow \theta = 0$

$y = 1,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \int_0^1 \int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_0^{\pi/2} \sin^2\theta\sqrt{1-\sin^2\theta}.\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\sqrt{\cos^2\theta}.\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{(2-1)(2-1)}{(2+2)(2+2-2)} \times \frac{\pi}{2}$$

$$\left[\because \int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots}.\frac{\pi}{2} \quad \text{if } m, n \text{ are even}\right]$$

$$= \frac{1(1)}{4(2)} \times \frac{\pi}{2} = \frac{\pi}{16}$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \frac{\pi}{16}$$

Q42. Change the order of integration in $\int_0^2 \int_0^{\sqrt{4-y^2}}$ xy dxdy and evaluate it.

Answer :

Given integral is,

$$\int_0^2 \int_0^{\sqrt{4-y^2}} xydxdy$$

y varies from 0 to 2

x varies from 0 to $\sqrt{4-y^2}$

$\Rightarrow \quad x = \sqrt{4-y^2}$

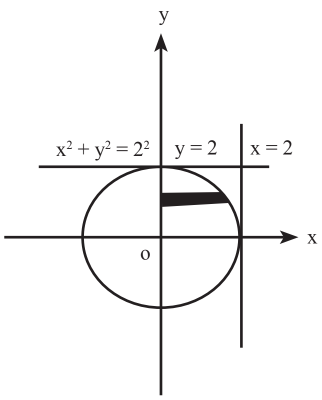
$\Rightarrow \quad x^2 = 4 - y^2$

$\Rightarrow \quad x^2 + y^2 = 4$

$\Rightarrow \quad x^2 + y^2 = 2^2$

The above equation represents a circle with centre at origin and radius 2 units.

Consider a strip parallel to X-axis as shown in figure.



Figure

$x^2 + y^2 = 4$

$y = 0$

$\Rightarrow \quad x = 2$

x varies from 0 to 2

$y^2 = 4 - x^2$

$\Rightarrow \quad y = \sqrt{4-x^2}$

$\therefore \quad y$ varies from 0 to $\sqrt{4-x^2}$

$$\therefore \int_0^2 \int_0^{\sqrt{4-y^2}} xydxdy = \int_0^2 \int_0^{\sqrt{4-x^2}} xydxdy$$

$$= \int_0^2 x\left[\frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \frac{1}{2}\int_0^2 x[4 - x^2 - 0]dx$$

$$= \frac{1}{2}\int_0^2 (4x - x^3)dx$$

$$= \frac{1}{2}\left[4\frac{x^2}{2} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{1}{2}\left[2.2^2 - \frac{2^4}{4}\right]$$

$$= \frac{1}{2}[8-4]$$

$$= 2$$

$$\therefore \int_0^2 \int_0^{\sqrt{4-y^2}} xydxdy = 2$$

Q43. Evaluate integral $\int_0^1 \int_{x^2}^{2-x}$ xy dy dx by changing the order of integration.

Answer :

Given integral is,

$$I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$$

The region of integration is bounded by the lines $x = 0$, $x = 1$

and $y = x^2$, $y = 2 - x$

i.e., x varies from 0 to 1

$\Rightarrow \quad y$ varies from x^2 to $2 - x$

Dividing the region of integration into two sub regions I_1 and I_2 as illustrated in figure 1

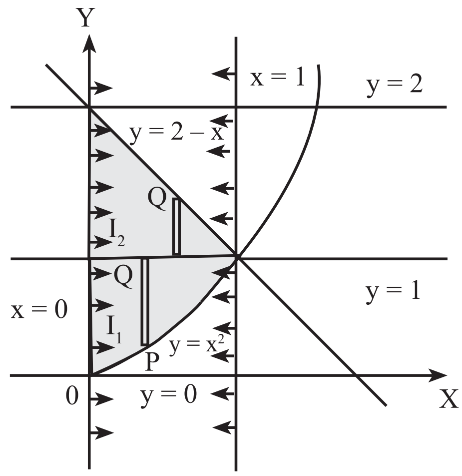


Figure (1)

i.e., $I = I_1 + I_2$ ... (1)

In the region I_1,

x varies from 0 to 1

y varies from x^2 to 1

Changing the order of integration, keeping y fixed

$$y = x^2$$

$$\Rightarrow \quad x = \sqrt{y}$$

x varies from 0 to $\sqrt{y}$

and y varies from 0 to 1

$\therefore$ Figure 2 represents the region of integration after changing the order.

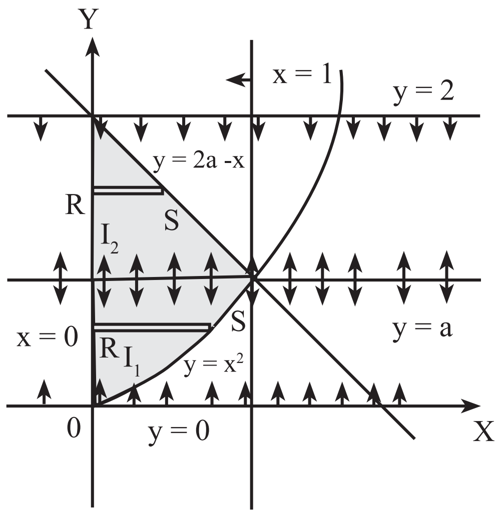


Figure (2)

$$\therefore \quad I_1 = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{y}} xy\, dx\, dy$$

$$= \int_0^1 \left[\int_0^{\sqrt{y}} x\, dx \right] y\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} y\, dy$$

$$= \int_0^1 \left[\frac{y}{2} - 0 \right] y\, dy$$

$$= \int_0^1 \frac{y^2}{2} dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - 0 \right] = \frac{1}{6}$$

$$\therefore \quad I_1 = \frac{1}{6} \qquad \text{... (2)}$$

In the region I_2,

x varies from 0 to 1

y varies from 1 to $2 - x$

Changing the order of integration, keeping y fixed,

$$y = 2 - x$$

$$\Rightarrow \quad x = 2 - y$$

$\therefore$ x varies from 0 to $2 - y$

Since $x = 0 \Rightarrow y = 2 - 0 = 2$

and $x = 1 \Rightarrow y = 2 - 1 = 1$

$\therefore$ y varies from 1 to 2

$$I_2 = \int_{y=1}^{2} \int_{x=0}^{2-y} xy\, dx\, dy$$

$$= \int_1^2 \left[\int_0^{2-y} x\, dx \right] y\, dy$$

$$= \int_1^2 \left[\frac{x^2}{2} \right]_0^{2-y} y\, dy$$

$$= \frac{1}{2}\int_1^2 [(2-y)^2 - 0]\, y\, dy$$

$$= \frac{1}{2}\int_1^2 [(4 - y^2 - 4y) y\, dy$$

$$= \frac{1}{2}\int_1^2 (4y + y^3 - 4y^2)\, dy$$

$$= \frac{1}{2}\left[4\int_1^2 y dy + \int_1^2 y^3 dy - 4\int_1^2 y^2 dy \right]$$

$$= \frac{1}{2}\left[4\left[\frac{y^2}{2} \right]_1^2 + \left[\frac{y^4}{4} \right]_1^2 - 4\left[\frac{y^3}{3} \right]_1^2 \right]$$

$$= \frac{1}{2}\left[2(4-1] + \frac{1}{4}[16-1] - \frac{4}{3}[8-1] \right]$$

$$= \frac{1}{2}\left[6 + \frac{15}{4} - \frac{28}{3} \right]$$

$$= \frac{1}{2}\left[\frac{5}{12} \right]$$

$$= \frac{5}{24}$$

$$\therefore \quad I_2 = \frac{5}{24} \qquad \text{... (2)}$$

Substituting equations (2) and (3) in equation (1),

$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \frac{1}{6} + \frac{5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}$$

$$\therefore \quad \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \frac{3}{8}$$

Q44. Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dy\,dx$ and hence evaluate it.

Answer :

Given integral is,

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dy\,dx$$

$$\Rightarrow \int_{y=0}^{y=a} \int_{x=y}^{x=a} \frac{x}{x^2+y^2}\,dx\,dy$$

The region of integration is bounded by the lines $x = y$, $x = a$ and $y = 0$, $y = a$ as illustrated in figure 1.

i.e., x varies from y to a

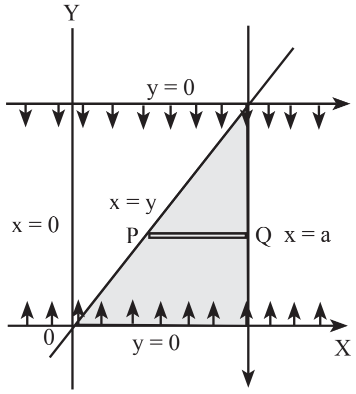


Figure (1)

Figure 2 represents the region of integration after changing the order.

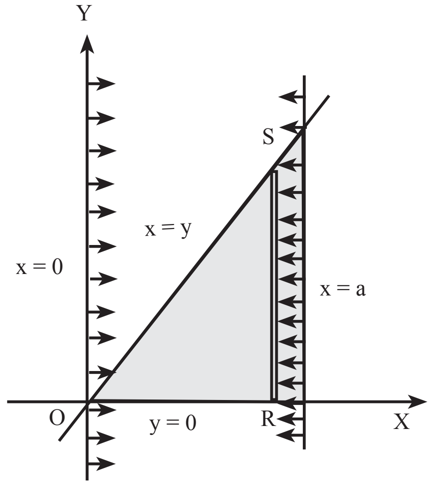


Figure (2)

Changing the order of integration,

x varies from 0 to a

Since $x = y$

$\therefore$ y varies from 0 to x

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \int_{x=0}^{x=a} \int_{y=0}^{y=x} \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a \left[\int_0^x \frac{x}{x^2+y^2}\,dy\right] dx$$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \int_0^a \left[\frac{x}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^a \left[\tan^{-1}\left(\frac{x}{x}\right) - \tan^{-1}(0)\right] dx$$

$$= \int_0^a (\tan^{-1}(1) - 0)\,dx$$

$$= \frac{\pi}{4}\int_0^a dx$$

$$= \frac{\pi}{4}[x]_0^a$$

$$= \frac{\pi}{4}[a-0]$$

$$= \frac{\pi}{4}a$$

$$\therefore \quad \int_0^a \int_y^a \frac{x}{x^2+y^2}\,dy\,dx = \frac{\pi}{4}a$$

4.3 CHANGE OF VARIABLES FROM CARTESIAN TO PLANE POLAR COORDINATES

Q45. Discuss about the change of variables from Cartesian to polar coordinates.

Answer :

Transformation of Coordinates

Double integrals can be easily evaluated by transforming the given integral into a simpler integral (of new variables) in an appropriate coordinate system.

Let, $x = f(u, v)$ and $y = g(u, v)$

Where,

x, y– Old variables

u, v – New variables.

Then,

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv \quad \text{... (1)}$$

Where,

J = Jacobian of the coordinate transformation

$$= \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Change of Variables from Cartesian to Polar Coordinates

In order to change a variable from Cartesian to polar coordinates, let

$u = r,\ v = \theta$ and

$x = r\cos\theta,\ y = r\sin\theta$

$\therefore$ Jacobian of transformation $= \dfrac{\partial(x,y)}{\partial(r,\theta)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta) = r(1) = r$

$\therefore\quad J = r$

Substituting corresponding values in equation (1),

$$\iint_R F(x,y)dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)r.dr.d\theta$$

$$\therefore\quad \iint F(r,\theta)dA = \int_{\theta=\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)rdr.d\theta$$

$\therefore$ In order to change variables from Cartesian to polar coordinates, the substitutions to be made are,

$x = r\cos\theta,\ y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.

Q46. By transforming into polar co-ordinates evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$. Hence find the value of $\int_0^\infty e^{-x^2}dx$.

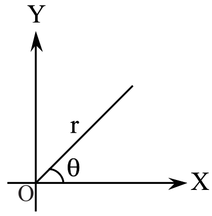
Answer :

Given integral is,

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy \qquad ...(1)$$

Equation (1) can be written in standard form as $\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)}dx\,dy$.

$\therefore$ Both x and y vary from 0 to ∞ and the region of integration is the first quadrant as shown in figure.



Figure

Let,

$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow\quad dx\,dy = r\,dr\,d\theta$

And $x^2 + y^2 = r^2$

$\therefore$ Limits are from $r = 0$ to ∞ and $\theta = 0$ to $\dfrac{\pi}{2}$ first quadrant

$$\therefore\quad \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=\infty} e^{-r^2} r\,dr\,d\theta \qquad ...(2)$$

Let, $r^2 = z$

$\Rightarrow\ 2\,rdr = dz$

Limits

For $r = 0, \Rightarrow z = 0$

For $r = \infty \Rightarrow z = \infty$

$\therefore$ Limits are from z = 0 to ∞.

Corresponding values in equation (2),

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_{\theta=0}^{\theta=\pi/2}\left[\frac{1}{2}\int_{z=0}^{z=\infty} e^{-z}dz\right]d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\theta=\pi/2}[e^{-z}]_{z=0}^{\infty}\ d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2}(e^{-\infty} - e^{-0})d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2}(0-1)d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}d\theta$$

$$= \frac{1}{2}[\theta]_{\theta=0}^{\pi/2} = \frac{\pi}{4}$$

$$\therefore\quad \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \frac{\pi}{4} \qquad ...(3)$$

Equation (1) can also be expressed as,

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy$$

$$= \left[\int_0^\infty e^{-x^2}dx\right]^2 \qquad ...(4)$$

Comparing equations (3) and (4)

$$\Rightarrow \left[\int_0^\infty e^{-x^2}dx\right]^2 = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Q47. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dydx$ by transforming to polar coordinates.

Answer :

Given integral is,

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$$

The region of integration is obtained as,

$y = 0, y = \sqrt{a^2-x^2}$

$\Rightarrow y^2 = a^2 - x^2$

$\Rightarrow x^2 + y^2 = a^2$

$\Rightarrow x = 0$ and $x = a$

The region is a quadrant of a circle $x^2 + y^2 = a^2$.

Let, $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = a^2$

$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = a^2$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = a^2$

$\Rightarrow r = a$

$dxdy = r\,dr\,d\theta$

r varies from 0 to a

θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dy\,dx = \int_{r=0}^{a}\int_{\theta=0}^{\frac{\pi}{2}} (r\sin\theta)r(rdrd\theta)$$

$$= \int_0^a\int_0^{\frac{\pi}{2}} r^3\sin\theta drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\left[\int_0^a r^3dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin\theta[a^4 - 0]d\theta$$

$$= \frac{a^4}{4}[-\cos\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{a^4}{4}[-[0-1]] = \frac{a^4}{4}$$

$$= \frac{a^4}{4}$$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dydx = \frac{a^4}{4}$$

Q48. Evaluate the following integral by transforming into polar coordinates $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$

Answer :

Given integral is,

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$$

The region of integration is bounded by the lines

$y = 0, y = \sqrt{a^2-x^2}$

$\Rightarrow y^2 = a^2 - x^2$

$\Rightarrow x^2 + y^2 = a^2$

and $x = 0, x = a$

Here the region is the quadrant of a circle $x^2 + y^2 = a^2$

Changing to polar coordinates

Let $x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

and $dx\,dy = r\,dr\,d\theta$

Limits for r : 0 to a

Limits for θ : 0 to $\frac{\pi}{2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^a (r\sin\theta)\sqrt{r^2}\,(rdrd\theta)$$

$$= \int_0^{\frac{\pi}{2}}\int_0^a (r\sin\theta)\,r\,rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a r^3 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^a r^3 \, dr \right] \sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_0^a \sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{a^4}{4} - \frac{0}{4} \right] \sin\theta \, d\theta$$

$$= \frac{a^4}{4} \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta$$

$$= \frac{a^4}{4} [-\cos\theta]_0^{\frac{\pi}{2}}$$

$$= -\frac{a^4}{4} \left[\cos\left(\frac{\pi}{2}\right) - \cos(0) \right]$$

$$= -\frac{a^4}{4} [0 - 1]$$

$$= -\frac{a^4}{4} (-1)$$

$$= \frac{a^4}{4}$$

$$\therefore \quad \int_0^a \int_0^{\sqrt{a^2 - x^2}} y\sqrt{x^2 + y^2} \, dx \, dy = \frac{a^4}{4}.$$

Q49. Evaluate by changing to polar coordinates

$$\int_0^a \int_y^a \frac{x}{x^2 + y^2} \, dx \, dy$$

Answer :

Given integral is,

$$\int_0^a \int_y^a \frac{x}{x^2 + y^2} \, dx \, dy$$

The region of integration is bounded by the planes $y = 0, y = a, x = y, x = a$

Let, $x = r\cos\theta$

$y = r\sin\theta$

and $dx\,dy = r dr\,d\theta$

If $x = a$, then

$a = r\cos\theta$

$\Rightarrow \quad r = \dfrac{a}{\cos\theta}$

$\Rightarrow \quad r = a\sec\theta$

If $x = 0$, then

$0 = r\cos\theta$

$\Rightarrow \quad r = 0$

$\therefore \quad r$ varies from 0 to $a\sec\theta$

If $y = 0$, then

$0 = r\sin\theta$

$\Rightarrow \quad \theta = 0$

Since $x = y$

$r\cos\theta = r\sin\theta$

$\cos\theta = \sin\theta$

$\Rightarrow \quad \theta = \dfrac{\pi}{4}$

$\therefore \quad \theta$ varies from 0 to $\dfrac{\pi}{4}$

$$\therefore \quad \int_0^a \int_y^a \frac{x}{x^2 + y^2} dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r\cos\theta \, r dr d\theta}{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r^2\cos\theta}{r^2} dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \cos\theta \, dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos\theta [r]_0^{a\sec\theta} \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos\theta [a\sec\theta - 0] d\theta$$

$$= \int_0^{\frac{\pi}{4}} a\,d\theta$$

$$= a[\theta]_0^{\frac{\pi}{4}}$$

$$= a\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{a\pi}{4}$$

$$\therefore \quad \int_0^a \int_y^a \frac{x}{x^2 + y^2} \, dx\,dy = \frac{a\pi}{4}$$

Q50. Evaluate $\iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$ over the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by using the transformation x = au and y = bv.

Answer :

Given integral is,

$$\iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$$

Ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

and transformation $x = au$, $y = bv$

Substituting the corresponding values in equation (1),

$\Rightarrow \quad \frac{a^2u^2}{a^2}+\frac{b^2v^2}{b^2}=1$

$\Rightarrow \quad u^2 + v^2 = 1$

$\therefore$ The region R is transformed to R'

Here R' is the area of circle $u^2 + v^2 = 1$ in the first quadrant

$\because \quad x = au, y = bv$

$$\frac{\partial x}{\partial u}=a\ ,\ \frac{\partial y}{\partial v}=b$$

and $\frac{\partial x}{\partial v}=0\ ,\ \frac{\partial y}{\partial u}=0$

The Jacobian of transformation is given by,

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$$

$$J = ab$$

$$\therefore \iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy = \iint_R (1-u^2-v^2)\,|J|\,du\,dv$$

$$= \iint_{R'} (1-u^2-v^2)\,ab\,du\,dv$$

Let $u = r\cos\theta$

$v = r\sin\theta$

and $du\,dv = r\,dr\,d\theta$

Limits for θ

If $u = 0$

$\Rightarrow \quad \cos\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

$\therefore \quad \theta$ varies from 0 to $\frac{\pi}{2}$

Limits for r

Since $u^2 + v^2 = 1$

$\Rightarrow \quad r = 1$

$\therefore \quad r$ varies from 0 to 1

$$\Rightarrow \iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1}(1-(u^2+v^2))abrdr\,d\theta$$

$$= ab\int_0^{\frac{\pi}{2}}\int_0^1 (1-r^2)r\,dr\,d\theta$$

$$= ab\int_0^{\frac{\pi}{2}}\int_0^1 (r-r^3)dr\,d\theta$$

$$= ab\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta$$

$$= ab\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}-\frac{1}{4}\right]d\theta$$

$$= ab\int_0^{\frac{\pi}{2}}\frac{1}{4}\,d\theta$$

$$= \frac{ab}{4}\int_0^{\frac{\pi}{2}}d\theta$$

$$= \frac{ab}{4}[\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{ab}{4}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi ab}{8}$$

$$\therefore \iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy = \frac{\pi ab}{8}.$$

Q51. Evaluate $\int\int_R \frac{xydxdy}{\sqrt{x^2+y^2}}$ where R is the region in the first quadrant enclosed by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.

Answer :

Given integral is,

$$I=\int\int_R \frac{xydxdy}{\sqrt{x^2+y^2}} \quad ... (1)$$

The circles are

$x^2 + y^2 = 4$ ... (2)

$x^2 + y^2 = 16$ ... (3)

Equation (2) represents a circle with centre at origin and radius 2 units

$\Rightarrow \quad x^2 + y^2 = 2^2$

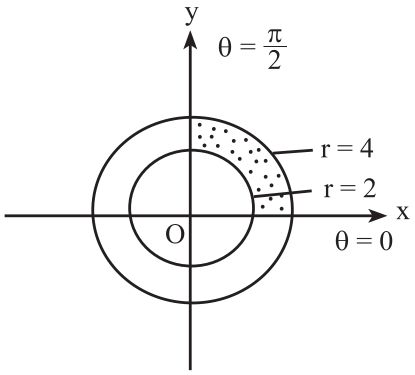
$\Rightarrow \quad r = 2$

Equation (2) represents a circle with centre at origin at radius 4 units

$\Rightarrow \quad x^2 + y^2 = 4^2$

$\Rightarrow \quad r = 4$

The region of integration, R is illustrated in figure.



Figure

Let, $x = r\cos\theta$, $y = r\sin\theta$

$dxdy = rdrd\theta$

r varies from 2 to 4

θ varies from 0 to $\frac{\pi}{2}$

Substituting the corresponding values in equation (1),

$$I = \iint_R \frac{xydxdy}{\sqrt{x^2+y^2}} = \int_0^{\frac{\pi}{2}}\int_2^4 \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}} r.\,dr.\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_2^4 \frac{r^3\cos\theta\sin\theta}{\sqrt{r^2(1)}} dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_2^4 \frac{r^3 2\cos\theta\sin\theta}{r} dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_2^4 r^2\sin 2\theta\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\left[\frac{r^3}{3}\right]_2^4 d\theta$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \sin 2\theta\,[(4)^3-(2)^3]\,d\theta$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \sin 2\theta\,[(64-8]\,d\theta$$

$$= \frac{56}{6}\int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta$$

$$= \frac{28}{3}\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{14}{3}\left[\cos 2\left(\frac{\pi}{2}\right) - \cos 0\right]$$

$$= -\frac{14}{3}[\cos\pi - 1]$$

$$= -\frac{14}{3}[-2]$$

$$= \frac{28}{3}$$

$$\therefore \quad \iint_R \frac{xy}{\sqrt{x^2+y^2}} dxdy = \frac{28}{3}$$

Q52. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dydx$ by changing into polar coordinates.

Answer :

Given integral is,

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$$

The region of integration is bounded by the lines $y = 0$, $y = \sqrt{2x-x^2}$ and $x = 0$, $x = 2$ as illustrated in figure 1.

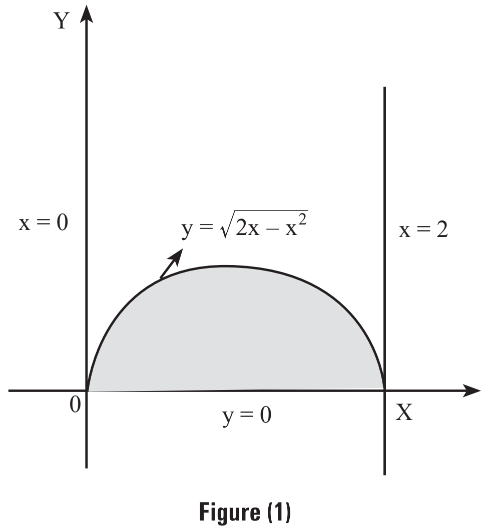


Figure (1)

$y = \sqrt{2x-x^2}$

$\Rightarrow \quad y^2 = 2x - x^2$

$\Rightarrow \quad x^2 + y^2 = 2x \qquad \text{... (1)}$

$y = 0$

$\Rightarrow \quad x^2 = 2x$

$\Rightarrow \quad x = 2$

$x = 0 \quad \Rightarrow y = 0$

Let, $x = r\cos\theta$

$y = r\sin\theta$

Substituting the corresponding values in equation (1),

$(r\cos\theta)^2 + (r\sin\theta)^2 = 2\, r\cos\theta$

$\Rightarrow \quad r^2(1) = 2\cos\theta$

$\Rightarrow \quad r = 2\cos\theta$

$\therefore \quad r$ varies from 0 to $2\cos\theta$

$x = 0 \Rightarrow r\cos\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

$y = 0 \Rightarrow r\sin\theta = 0$

$\Rightarrow \quad \theta = 0$

$\therefore \quad \theta$ varies from 0 to $\frac{\pi}{2}$

The region of integration is bounded by the curves $r = 0$, $r = 2\cos\theta$ and $\theta = 0$, $\theta = \frac{\pi}{2}$ in polar coordinates as illustrated in figure 2.

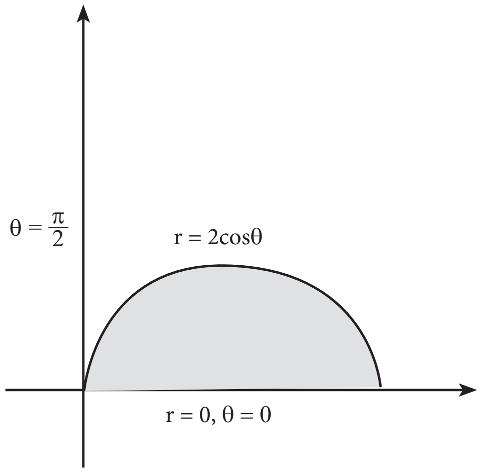


Figure (2)

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r\cos\theta)^2 + (r\sin\theta)^2 r.dr d\theta$$

$[\because dx\,dy = r\,dr\,d\theta]$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2(1)\, r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\, dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(2\cos\theta)^4}{4} d\theta$$

$$= \int_0^{\frac{\pi}{2}} 16\frac{\cos^4\theta}{4} d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 4.\frac{4-1}{4}.\frac{4-3}{4-2}.\frac{\pi}{2}$$

$$\left[\because \int_0^{\frac{\pi}{2}} \cos^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}....\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= 4.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi}{4}$$

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx = \frac{3\pi}{4}$$

Q53. Transform the integral into polar coordinates and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$.

Answer :

Given integral is,

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

The region of integration is bounded by the planes

$x = 0$, $x = a$ and $y = 0$, $y = \sqrt{a^2-x^2}$

$\Rightarrow \quad y^2 = a^2 - x^2$

$\Rightarrow \quad x^2 + y^2 = a^2$

Let, $x = r\cos\theta$

$y = r\sin\theta$

and $dx\,dy = r\,dr\,d\theta$

$r^2 = x^2 + y^2$

$\Rightarrow \quad \sqrt{x^2+y^2} = r$

If $x = 0$

$\Rightarrow \quad r\cos\theta = 0$

$\Rightarrow \quad \cos\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

$\therefore \quad \theta$ varies from 0 to $\frac{\pi}{2}$

and r varies from 0 to a

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^a r(r dr d\theta)$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^a r^2 dr\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{a^3}{3}\right] d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} d\theta = \frac{a^3}{3}[\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{a^3}{3}\left[\frac{\pi}{2} - 0\right] = \frac{\pi a^3}{6}$$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \frac{\pi a^3}{6}$$

Q54. Evaluate, by changing to polar coordinates the integral $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2} dx\, dy$.

Answer :

Given integral is,

$$\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dxdy$$

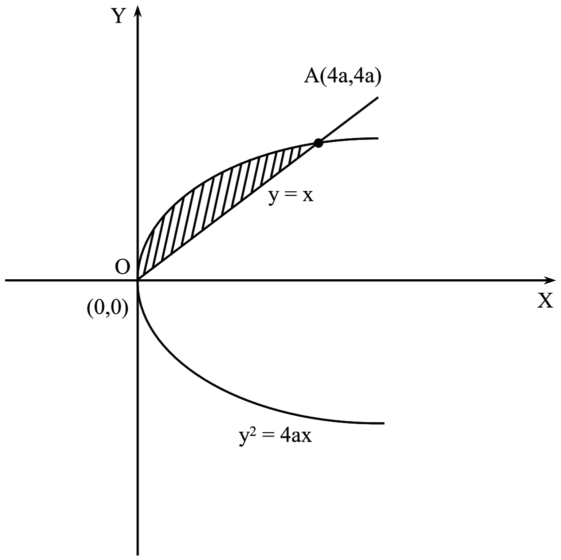
The region of integration is bounded by the parabola and the straight line

i.e., $x = \frac{y^2}{4a}, x = y$

$\Rightarrow \quad y^2 = 4ax, y = x$

'y' varies from 0 to $4a$

Therefore, the point of intersection of the parabola and straight is as shown in figure below.



Figure

Changing to Polar Coordinate

Let,

$x = r\cos\theta, y = r\sin\theta$

$\Rightarrow \quad dxdy = rdrd\theta$

Limits of r

$$x = \frac{y^2}{4a}$$

$$\Rightarrow r\cos\theta = \frac{r^2\sin^2\theta}{4a}$$

$$\Rightarrow \quad r = \frac{4a\cos\theta}{\sin^2\theta}$$

Limits of θ

Equation of the line is,

$y = x$

Slope, $m = 1 \quad [\because y = mx]$

$m = \tan\theta$

$\Rightarrow \quad 1 = \tan\theta$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$\therefore \quad \theta$ varies from, $\frac{\pi}{4}$ to $\frac{\pi}{2}$

$$\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2} dxdy = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{(r\cos\theta)^2-(r\sin\theta)^2}{r^2\cos^2\theta + r^2\sin^2\theta}.rdrd\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{\frac{4a\sin\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta)\, rdrd\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} .d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos^2\theta - \sin^2\theta)\left(\frac{4a\cos\theta}{\sin^2\theta}\right)^2 d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos^2\theta - \sin^2\theta)\frac{\cos^2\theta}{\sin^4\theta} d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4\theta - \cot^2\theta) d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2\theta(\text{cosec}^2\theta - 1) - (\text{cosec}^2\theta - 1)\,d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\cot^2\theta\,\text{cosec}^2\theta - 2\,\text{cosec}^2\theta + 2)]\,d\theta$$

$$= 8a^2\left[-\frac{[\cot\theta]^{2+1}}{2+1} + 2\cot\theta + 2\theta\right]_{\pi/4}^{\pi/2}$$

$$\left(\because \int f^n(x) f'(x) dx = \frac{f(x)^{n+1}}{n+1} + c\right)$$

$$= 8a^2\left[-\frac{\left(\cot\frac{\pi}{2}\right)^3}{3} + 2\cot\frac{\pi}{2} + \frac{2\pi}{2} + \frac{\left(\cot\frac{\pi}{4}\right)^3}{3} - 2\cot\frac{\pi}{4} - 2\frac{\pi}{4}\right]$$

$$= 8a^2\left[\pi + \frac{1}{3} - 2 - \frac{\pi}{2}\right]$$

$$= 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

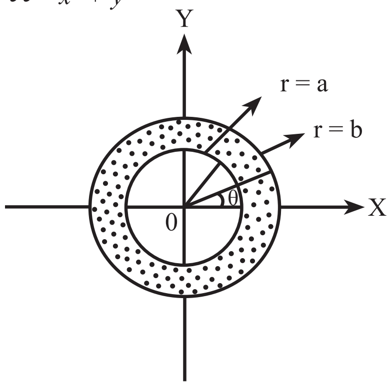
$$\therefore \int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2} dxdy = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

Q55. By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2} dx\,dy$ over the annular region between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 + b^2$ (b > a).

Answer :

Given integral is,

$$\iint \frac{x^2y^2}{x^2 + y^2} dx\,dy$$



Figure

And circles $x^2 + y^2 = a^2$,

$x^2 + y^2 = b^2$

Changing to polar coordinates

Let, $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow$ $dx\,dy = r dr d\theta$

Then $x^2 + y^2 = a^2$,

$\Rightarrow$ $r^2\cos^2\theta + r^2\cos^2\theta = a^2$

$\Rightarrow$ $r^2(\cos^2\theta + \sin^2\theta) = a^2$

$\Rightarrow$ $r^2 = a^2$

$\Rightarrow$ $r = a$

Similarly, $x^2 + y^2 = b^2$

$\Rightarrow$ $r^2 = b^2$

$\Rightarrow$ $r = b$

$\therefore$ r varies from a to b

and θ varies from 0 to 2π.

$$\therefore \iint \frac{x^2y^2}{x^2+y^2} dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\cos^2\theta . r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} r dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b \frac{r^5\cos^2\theta\sin^2\theta}{r^2(\cos^2\theta + \sin^2\theta)} dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_a^b r^3\,dr\right]\cos^2\theta\sin^2\theta\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_a^b \cos^2\theta\sin^2\theta\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{b^4}{4} - \frac{a^4}{4}\right]\cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{b^4 - a^4}{4}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta$$

[Multiply and divide by 4]

$$= \frac{b^4 - a^4}{16}\int_0^{2\pi} 4\cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{b^4 - a^4}{16}\int_0^{2\pi}\sin^2 2\theta\,d\theta$$

$$= \frac{b^4 - a^4}{16} \int_0^{2\pi} \left(\frac{1 - \cos 4\theta}{2}\right) d\theta$$

$$= \frac{b^4 - a^4}{32} \int_0^{2\pi} (1 - \cos 4\theta) d\theta$$

$$= \frac{b^4 - a^4}{32} \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{b^4 - a^4}{32} \left[(2\pi - 0) - \left(\frac{\sin 4(2\pi)}{4} - \frac{\sin 4(0)}{4}\right)\right]$$

$$= \frac{b^4 - a^4}{32} [2\pi - (0 - 0)]$$

$$= \frac{\pi}{16} (b^4 - a^4)$$

$$\therefore \quad \iint \frac{x^2 y^2}{x^2 + y^2} \, dx \, dy = \frac{\pi}{16} (b^4 - a^4)$$

4.4 TRIPLE INTEGRALS

Q56. Write a short note on Tripe Integrals.

Answer :

Triple Integrals

The generalized integration of a definite integral to three dimensions is known as triple integral. In this case, the definite integral of a single variable function is extended to a function of three variables.

Explanation

Let,

V = 3 dimensional finite region

$f(x, y, z)$ = Function which is defined over 'V'

$\delta V_1, \delta V_2, ..., \delta V_n$ = 'n' elementary sub-divisions or volumes in V

(x_r, y_r, z_r) = A point in r^{th} sub division, δV_r.

Consider the sum, $\sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$ ... (1)

The finite limit of the sum (equation (1)) for $n \to \infty$ and $\delta V_r \to 0$ is termed as triple integral of $f(x, y, z)$ over the region V. It is represented by,

$$\iiint f(x, y, z) dV .$$

Q57. Evaluate $\int_1^{e} \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

Answer :

Given integral is,

$$\int_1^{e} \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$$

The above integral can be evaluated as,

$$\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy = \int_1^e \int_1^{\log y} \left[\int_1^{e^x} \log z \, dz \right] dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[\log z \int 1 dz - \int \frac{d}{dz} \log z \int 1 dz \right]^{e^x} dxdy$$

$$= \int_1^e \int_1^{\log y} \left[\log z(z) - \int \frac{1}{z} \times z dz \right]_1^{e^x} dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[\log z(z) - \int 1 \, dz \right]_1^{e^x} dx \, dy$$

$$= \int_1^e \int_1^{\log y} [z \log z - z]_1^{e^x} dx \, dy$$

$$= \int_1^e \int_1^{\log y} [e^x \log e^x - e^x] - [1 \log 1 - 1] \, dx \, dy$$

$$= \int_1^e \int_1^{\log y} [e^x \log e^x - e^x] - [0 - 1] \, dx \, dy$$

$$= \int_1^e \int_1^{\log y} [e^x \log e^x - e^x + 1] \, dx \, dy$$

$$= \int_1^e \int_1^{\log x} [e^x . x \log e_e - e^x + 1] \, dx \, dy$$

$$= \int_1^e \int_1^{\log y} [xe^x - e^x + 1] \, dx \, dy$$

$$= \int_1^e \left[\int_1^{\log y} xe^x dx - \int_1^{\log y} e^x dx + \int_1^{\log y} 1 dx \right] dy$$

$$= \int_1^e \left[\left[x \int e^x dx - \int \frac{dx}{dx} \int e^x dx \right] - e^x + x \right]_1^{\log y} dy$$

$$= \int_1^e \left[xe^x - \int e^x dx \right] - e^x + x \Big]_1^{\log y} dy$$

$$= \int_1^e \left[xe^x - e^x - e^x + x \right]_1^{\log y} dy$$

$$= \int_1^e \left[xe^x - 2e^x + x \right]_1^{\log y} dy$$

$$= \int_1^e \left[e^x (x - 2) + x \right]_1^{\log y} dy$$

$$= \int_1^e \left[e^{\log y}(\log y - 2) + \log y - [e(1-2)+1]\right] dy$$

$$= \int_1^e \left[e^{\log y}(\log y - 2) + \log y - [e(-1)+1]\right] dy$$

$$= \int_1^e \left[e^{\log y}(\log y - 2) + \log y + e - 1\right] dy$$

$$= \int_1^e \left[y\log y - 2y + \log y + e - 1\right] dy \qquad [\because e^{\log_e^x} = x]$$

$$= \int_1^e [(y+1)\log y - 2y + e - 1] dy$$

$$= \left[\left[\log y \int (y+1)dy - \int \frac{d}{dy}\log y \int (y+1)dy\right] - 2\int y dy + e\int dy - \int dy\right]_1^e$$

$$= \left[\left[\log y\left(\frac{y^2}{2} + y\right) - \int\left(\frac{y^2}{2} + y\right)\frac{1}{y}dy\right] - \frac{2y^2}{2} + ey - y\right]_1^e$$

$$= \left[\log y\left[\frac{y^2}{2} + y\right] - \int\left(\frac{y}{2} + 1\right)dy - y^2 + ey - y\right]_1^e$$

$$= \left[\log y\left[\frac{y^2}{2} + y\right] - \frac{y^2}{4} - y - y^2 + ey - y\right]_1^e$$

$$= \left[\log y\left[\frac{y^2}{2} + y\right] - \frac{5y^2}{4} - 2y + ey\right]_1^e$$

$$= \left[\log e\left[\frac{e^2}{2} + e\right] - \frac{5e^2}{4} - 2e + e.e\right] - \left[\log 1\left[\frac{1}{2} + 1\right] - \frac{5}{4} - 2 + e\right]$$

$$= \left[1\left[\frac{e^2}{2} + e\right] - \frac{5e^2}{4} + e^2 - 2e\right] - \left[0 - \frac{13}{4} + e\right] = \left[\frac{e^2}{2} + e - \frac{e^2}{4} - 2e\right] - \left[\frac{-13}{4} + e\right]$$

$$= \left[\frac{2e^2 - e^2}{4} - e\right] + \frac{13}{4} - e$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

$$= \frac{1}{4}(e^2 - 8e + 13)$$

$$\therefore \quad \int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy = \frac{1}{4}(e^2 - 8e + 13)$$

Q58. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y}$ dx dy dz.

Answer :

Given integral is,

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\, dy\, dz$$

The above integral can be evaluated as,

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\, dy\, dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left\{ \int_0^{1-x-y} dz \right\} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \{z\}_0^{1-x-y}\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \{1-x-y-0\} dy\, dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} (1-x-y) dy \right\} dx$$

$$= \int_0^1 \left\{ y - xy - \frac{y^2}{2} \right\}_0^{1-x} dx$$

$$= \int_0^1 \left\{ (1-x) - x(1-x) - \frac{(1-x)^2}{2} - 0 \right\} dx$$

$$= \int_0^1 \left\{ 1-x-x+x^2 - \left(\frac{1+x^2-2x}{2} \right) \right\} dx$$

$$= \int_0^1 \left\{ 1-x-x+x^2 - \frac{1}{2} - \frac{x^2}{2} + x \right\} dx$$

$$= \int_0^1 \left\{ 1-x+x^2 - \frac{1}{2} - \frac{x^2}{2} \right\} dx$$

$$= \int_0^1 \left\{ \frac{2-2x+2x^2-1-x^2}{2} \right\} dx$$

$$= \int_0^1 \left\{ \frac{1-2x+x^2}{2} \right\} dx$$

$$= \frac{1}{2} \left\{ \int_0^1 (1-2x+x^2) \right\} dx$$

$$= \frac{1}{2} \left\{ x - \frac{2x^2}{2} + \frac{x^3}{3} \right\}_0^1$$

$$= \frac{1}{2} \left\{ 1 - \frac{2(1)^2}{2} + \frac{(1)^3}{3} - 0 \right\}$$

$$= \frac{1}{2} \left\{ 1 - \frac{2}{2} + \frac{1}{3} \right\}$$

$$= \frac{1}{2} \left\{ 1 - 1 + \frac{1}{3} \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{3} \right\} = \frac{1}{6}$$

$$\therefore \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\, dy\, dz = \frac{1}{6}.$$

Q59. Evaluate $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2) dx\, dy\, dz$.

Answer :

Given integral is,

$$\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2) dx\, dy\, dz$$

The above integral can be evaluated as,

$$\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2) dx\, dy\, dz$$

$$= \int_0^a \int_0^b \left[\int_0^c (x^2+y^2+z^2) dz \right] dx\, dy$$

$$= \int_0^a \int_0^b \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_0^c dx\, dy$$

$$= \int_0^a dx \int_0^b \left[cx^2 + cy^2 + \frac{c^3}{3} \right] dy$$

$$= \int_0^a dx \int_0^b \left(cx^2 + cy^2 + \frac{c^3}{3} \right) dy$$

$$= \int_0^a \left[cx^2 y + \frac{cy^3}{3} + \frac{c^3 y}{3} \right]_0^b dx$$

$$= \int_0^a \left[x^2 bc + \frac{b^3 c}{3} + \frac{bc^3}{3} \right] dx$$

$$= \int_0^a \left(x^2 bc + \frac{b^3 c}{3} + \frac{bc^3}{3} \right) dx$$

$$= \left[\frac{x^3}{3}bc + \frac{xb^3c}{3} + \frac{xbc^3}{3}\right]_0^a$$

$$= \frac{a^3bc}{3} + \frac{ab^3c}{3} + \frac{abc^3}{3}$$

$$= \frac{abc}{3}[a^2 + b^2 + c^2]$$

$$\therefore \int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz = \frac{abc}{3}[a^2+b^2+c^2].$$

Q60. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz$.

Answer :

Given integral is,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz$$

The above integral can be evaluated as,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz$$

$$= \int_0^1 x\,dx \int_0^{\sqrt{1-x^2}} y\,dy \int_0^{\sqrt{(1-x^2-y^2)}} z\,dz$$

$$= \int_0^1 x\,dx \int_0^{\sqrt{1-x^2}} \left[\frac{z^2}{2}\right]_0^{\sqrt{(1-x^2-y^2)}} y\,dy$$

$$= \int_0^1 x\,dx \int_0^{\sqrt{(1-x^2)}} y\,dy\left[\frac{\left(\sqrt{1-x^2-y^2}\right)^2}{2} - \frac{0}{2}\right]$$

$$= \int_0^1 x\,dx \int_0^{\sqrt{(1-x^2)}} \left[\frac{1-x^2-y^2}{2}\right] y\,dy$$

$$= \int_0^1 x\,dx \int_0^{\sqrt{(1-x^2)}} \left(y(1-x^2) - y^3\right)dy$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{\left(\sqrt{1-x^2}\right)^2}{2} - \frac{\left(\sqrt{1-x^2}\right)^4}{4}\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{(1-x^2)}{2} - \frac{(1-x^2)^2}{4}\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x^2)^2}{2} - \frac{(1-x^2)^2}{4}\right]dx$$

$$= \frac{1}{8}\int_0^1 x(1-x^2)^2\,dx$$

$$= \frac{1}{8}\int_0^1 x(x^4 - 2x^2 + 1)\,dx$$

$$= \frac{1}{8}\int_0^1 (x^5 - 2x^3 + x)\,dx$$

$$= \frac{1}{8}\left[\frac{x^6}{6} - \frac{2x^4}{4} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{8}\left[\left[\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right] - 0\right]$$

$$= \frac{1}{8}\cdot\frac{1}{6}$$

$$= \frac{1}{48}$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz = \frac{1}{48}$$

Q61. Evaluate $\iiint xy^2 z\,dx\,dy\,dz$ taken through the positive octant of the sphere: $x^2 + y^2 + z^2 = a^2$.

Answer :

Given integral is,

$$\int\int\int xy^2 z\,dx\,dy\,dz$$

Equation of sphere is,

$$x^2 + y^2 + z^2 = a^2$$

$$\Rightarrow z^2 = a^2 - x^2 - y^2$$

$$\Rightarrow z = \sqrt{a^2 - x^2 - y^2}$$

The limits of z are 0 to $\sqrt{a^2 - x^2 - y^2}$

The projection of sphere on XY-plane is a circle $x^2 + y^2 = a^2$

$$\Rightarrow y^2 = a^2 - x^2$$

$$y = \sqrt{a^2 - x^2}$$

$\therefore$ y varies from 0 to $\sqrt{a^2 - x^2}$

x varies from 0 to a

$$\iiint xy^2 z\,dxdydz = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xy^2 z\,dxdydz$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2 \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dxdy$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2 \left[\frac{a^2-x^2-y^2}{2} - 0\right] dxdy$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (xy^2 a^2 - x^3 y^2 - xy^4)\,dxdy$$

$$= \frac{1}{2}\int_0^a x\left[\int_0^{\sqrt{a^2-x^2}} \{(a^2-x^2)y^2 - y^4\}\,dy\right] dx$$

$$= \frac{1}{2}\int_0^a x\left[(a^2-x^2)\frac{y^3}{3} - \frac{y^5}{5}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\left[\frac{a^2-x^2}{3}(a^2-x^2)^{\frac{3}{2}} - \frac{(a^2-x^2)^{\frac{5}{2}} - 0}{5}\right] dx$$

$$= \frac{1}{2}\int_0^a x\left[\frac{(a^2-x^2)^{\frac{5}{2}}}{3} - \frac{(a^2-x^2)^{\frac{5}{2}}}{5}\right] dx = \frac{1}{2}\int_0^a x(a^2-x^2)^{\frac{5}{2}}\left[\frac{1}{3} - \frac{1}{5}\right] dx$$

$$= \frac{1}{2}\int_0^a x(a^2-x^2)^{\frac{5}{2}}\left(\frac{2}{15}\right) dx = \frac{1}{15}\int_0^a x(a^2-x^2)^{\frac{5}{2}}\,dx$$

Let, $a^2 - x^2 = t$

$\Rightarrow \quad -2x\,dx = dt$

$\Rightarrow \quad xdx = \frac{-dt}{2}$

$x = 0 \Rightarrow t = a^2$

$x = a^2 \Rightarrow t = 0$

$\therefore \quad t$ varies from 0 to a^2

$$\iiint xy^2 z\,dxdydz = \frac{1}{15}\int_0^a x(a^2-x^2)^{\frac{5}{2}}\,dx = \frac{1}{15}\int_{a^2}^0 (t)^{\frac{5}{2}}\left(\frac{-dt}{2}\right)$$

$$= \frac{-1}{30}\int_{a^2}^0 t^{\frac{5}{2}}\,dt$$

$$= \frac{-1}{30}\left[\frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right]_{a^2}^0$$

$$= -\frac{1}{30}\left[\frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right]_{a^2}^0$$

$$= -\frac{1}{30}\times\frac{2}{7}\left[0 - (a^2)^{\frac{7}{2}}\right]$$

$$= -\frac{1}{105}[-a^7] = \frac{a^7}{105}$$

$$\therefore \quad \iiint xy^2 z\,dxdydz = \frac{a^7}{105}$$

Q62. Evaluate $\iiint_V \frac{dzdydx}{(x+y+z+1)^3}$ where V is the region bounded by x = 0, y = 0, z = 0 and x + y + z = 1.

Answer :

Given integral is,

$$\iiint_V \frac{dz\,dydx}{(x+y+z+1)^3}$$

Here, V is region bounded by the planes

$x = 0, y = 0, z = 0, x + y + z = 1$

If $y = 0, z = 0$

$\Rightarrow$ $x = 1$

$\therefore$ x varies from 0 to 1

If $z = 0$

$\Rightarrow$ $x + y = 1$

$\Rightarrow$ $y = 1 - x$

$\therefore$ y varies from 0 to $1 - x$.

$\therefore$ $x + y + z = 1$

$\Rightarrow$ $z = 1 - x - y$

$\therefore$ z varies from 0 to $1 - x - y$.

$$\therefore \iiint_V \frac{1}{(x+y+z+1)^3}dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3}dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\int_0^{1-x-y}(x+y+z+1)^{-3}dz\right]dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y}dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[x+y+(1-x-y)+1\right]^{-2}-\left[x+y+0+1\right]^{-2}dy\,dx$$

$$= -\frac{1}{2}\int_0^{1-x}\left((2)^{-2}-(x+y+1)^{-2}\right)dy\,dx$$

$$= -\frac{1}{2}\int_0^1\left[\int_0^{1-x}\left[\frac{1}{4}-(x+y+1)^{-2}\right]dy\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4}[y]_0^{1-x}-\left[\frac{(x+y+1)^{-2+1}}{-2+1}\right]_0^{1-x}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1}{4}[1-x]-\left[\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x}\right]dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1}{4}(1-x)+[(x+(1-x)+1)^{-1}-(x+0+1)^{-1}\right]dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{1}{4} - \frac{1}{4}x + [2]^{-1} - (x+1)^{-1}\right)dx$$

$$= -\frac{1}{2}\left[\int_0^1 \frac{1}{4}dx - \int_0^1 \frac{1}{4}xdx + \int_0^1 \frac{1}{2}dx - \int_0^1 \frac{1}{(x+1)}dx\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4}[x]_0^1 - \frac{1}{4}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_0^1 - [\log(x+1)]_0^1\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4}[1-0] - \frac{1}{8}[1-0] + \frac{1}{2}[1-0] - [\log(1+1) - \log(1+0)]\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{8} + \frac{1}{2} - \log 2 + \log 1\right]$$

$$= -\frac{1}{2}\left[\frac{10}{16} - \log 2\right]$$

$$= -\frac{5}{16} + \frac{1}{2}\log 2$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$

$$\therefore \quad \iiint_V \frac{1}{(x+y+z+1)^3}\,dz\,dy\,dx = \frac{1}{2}\log 2 - \frac{5}{16}$$

Q63. Evaluate $\iiint (xyz)dz\,dy\,dz$ over the first octant of $x^2 + y^2 + z^2 = a^2$.

OR

Find the value of $\iiint (x\,y\,z)\,dx\,dy\,dz$ through the positive spherical octant for which $x^2 + y^2 + z^2 \leq a^2$

Answer :

Given integral is,

$$\iiint xyz\,dx\,dy\,dz$$

Equation of sphere is,

$$x^2 + y^2 + z^2 = a^2$$

$$\Rightarrow \quad z^2 = a^2 - x^2 - y^2$$

$$\Rightarrow \quad z = \sqrt{a^2 - x^2 - y^2}$$

$\therefore$ z varies from 0 to $\sqrt{a^2 - x^2 - y^2}$

Since, the given integral is on the positive octant of sphere

$\Rightarrow$ Equation of circle is $x^2 + y^2 = a^2$

$$\Rightarrow \quad y^2 = a^2 - x^2$$

$$\Rightarrow \quad y = \sqrt{a^2 - x^2}$$

$\therefore$ y varies from 0 to $\sqrt{a^2 - x^2}$

$$y = 0 \Rightarrow x = 0$$

and x varies from 0 to a

$$\therefore \iiint xyz\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\,dx\,dy\,dz$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\int_0^{\sqrt{a^2-x^2-y^2}} z\,dz\right]xy\,dx\,dy$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} xy\,dx\,dy$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\frac{a^2-x^2-y^2}{2}-0\right]xy\,dx\,dy$$

$$= \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[(a^2-x^2-y^2)y\,dy\right]x\,dx$$

$$= \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[(a^2y-x^2y-y^3)dy\right]x\,dx$$

$$= \frac{1}{2}\int_0^a\left[\frac{a^2y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} x\,dx$$

$$= \frac{1}{2}\int_0^a\left[\frac{a^2(a^2-x^2)}{2}-\frac{x^2}{2}(a^2-x^2)-\frac{1}{4}(a^2-x^2)^2-0\right]x\,dx$$

$$= \frac{1}{2}\int_0^a\left[\frac{a^4-a^2x^2)}{2}-\frac{(a^2x^2-x^4)}{2}-\frac{1}{4}(a^4+x^4-2a^2x^2)\right]x\,dx$$

$$= \frac{1}{2}\int_0^a\left[2\frac{(a^4-a^2x^2-a^2x^2+x^4)-(a^4+x^4-2a^2x^2)}{4}\right]x\,dx$$

$$= \frac{1}{8}\int_0^a(2a^4x-2a^2x^3-2a^2x^3+2x^5-a^4x+x^5+2a^2x^3)dx$$

$$= \frac{1}{8}\int_0^a(a^4x-2a^2x^3+x^5)dx$$

$$= \frac{1}{8}\left[a^4\int_0^a x\,dx-2a^2\int_0^a x^3\,dx+\int_0^a x^5\,dx\right]$$

$$= \frac{1}{8}\left[a^4\left[\frac{x^2}{2}\right]_0^a-2a^2\left[\frac{x^4}{4}\right]_0^a+\left[\frac{x^6}{6}\right]_0^a\right]$$

$$= \frac{1}{8}\left[\frac{a^4}{2}[a^2-0]-\frac{a^2}{2}[a^4-0]+\frac{1}{6}[a^6-0]\right]$$

$$= \frac{1}{8}\left[\frac{a^6}{2}-\frac{a^6}{2}+\frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\frac{a^6}{6}\right]$$

$$= \frac{a^6}{48}$$

$$\therefore \quad \iiint xyz\,dx\,dy\,dz = \frac{a^6}{48}$$

Q64. Find the volume of the cylinder $x^2 + y^2 = 25$ bounded by the planes $z = 1$ and $x + z = 10$.

Answer :

Given that,

Cylinder, $x^2 + y^2 = 25$ ... (1)

Planes, $z = 1$

$x + z = 10$ ... (2)

Volume of the cylinder is given as

$$V = \int\int_V\int dx\, dy\, dz \quad \text{... (3)}$$

From equation (2)

$z = 10 - x$

$\therefore$ z varies from 1 to $10 - x$

From equation (1),

$x^2 = 25 - y^2$

$\Rightarrow$ $x = \pm\sqrt{25 - y^2}$

$\therefore$ x varies from $-\sqrt{25 - y^2}$ to $\sqrt{25 - y^2}$

y varies from -5 to 5

Substituting the corresponding values in equation (3)

$$V = \int_{-5}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} \int_{1}^{10-x} dx\, dy\, dz$$

$$= \int_{-5}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} [z]_1^{10-x} dy\, dx$$

$$= \int_{-5}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} [10 - x - 1] dy\, dx$$

$$= \int_{-5}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} [9 - x] dy\, dx$$

$$= \int_{-5}^{5} \left[9x - \frac{x^2}{2}\right]_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dy$$

$$= \int_{-5}^{5} \left\{9\left[\sqrt{25-y^2} - \left(-\sqrt{25-y^2}\right)\right] - \left[\left(\sqrt{25-y^2}\right)^2 - \left(-\sqrt{25-y^2}\right)^2\right]\right\} dy$$

$$= \int_{-5}^{5} \left[9.2\sqrt{25-y^2} - \frac{(25-y^2) - (25-y^2)}{2}\right] dy$$

$$= 18 \int_{-5}^{5} \left(\sqrt{25-y^2}\right) dy$$

As $\sqrt{25-y^2}$ is an even function,

$$V = 18 \times 2 \int_0^5 \left(\sqrt{25-y^2}\right)dy$$

$$= 36\left[\frac{y\sqrt{25-y^2}}{2} + \frac{25}{2}\sin^{-1}\frac{y}{5}\right]_0^5$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \quad \text{Here, } a = 5\right]$$

$$= 36\left[\left[\frac{5\sqrt{25-5^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{5}{5}\right) - 0\right]\right]$$

$$= 36\left[0 + \frac{25}{2}\sin^{-1}(1)\right]$$

$$= 36.\frac{25}{2}.\frac{\pi}{2}$$

$$= 225\pi$$

$\therefore$ Volume of the cylinder = 225π cubic units.

Q65. Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Answer :

Given that,

Paraboloid, $z = x^2 + y^2$ ... (1)

Plane $z = 4$ ... (2)

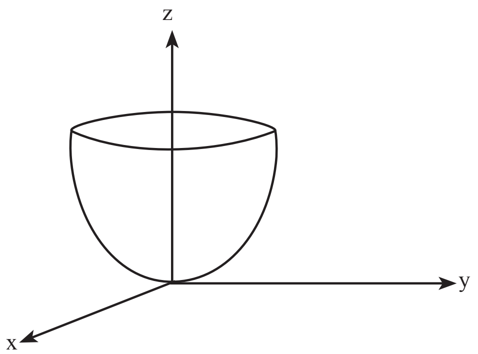
$\therefore$ z varies from $x^2 + y^2$ to 4

Comparing equations (1) and (2).

$x^2 + y^2 = 4$

The above equation represents a circle with center at origin and radius 2 units.

Figure illustrates the paraboloid.



Figure

The projection of region on xy plane gives the area of the circle $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = 0 \Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

$\therefore$ x varies from -2 to $+2$

$y = \pm\sqrt{4-x^2}$

$\therefore$ y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

The volume of the region is given as,

$$V = \iiint_V dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_{x^2+y^2}^{4}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+y^2)\right]dy\,dx$$

The integrand i.e., $4-(x^2+y^2)$ is even with respect to both x and y.

$$\therefore \quad V = 2 \times \int_0^2 2 \times \int_0^{\sqrt{4-x^2}} (4-(x^2+y^2))\,dy\,dx$$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}} [4-x^2-y^2]\,dy\,dx$$

$$= 4\int_0^2 \left[(4-x^2)y - \frac{y^3}{3}\right]_0^{\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \left[4-x^2\left[\sqrt{4-x^2}-0\right] - \frac{\left(\sqrt{4-x^2}\right)^3}{3} - 0\right]dx$$

$$= 4\int_0^2 \left[(4-x^2\sqrt{4-x^2} - \frac{(4-x^2)(\sqrt{4-x^2}}{3}\right]dx$$

$$= 4\int_0^2 \sqrt{4-x^2}\left[4-x^2-\frac{(4-x^2)}{3}\right]dx$$

$$= 4\int_0^2 \sqrt{4-x^2}\,\frac{2}{3}(4-x^2)\,dx$$

$$V = \frac{8}{3}\int_0^2 (4-x^2)^{\frac{3}{2}}\,dx \quad \text{... (3)}$$

Let,

$x = 2\sin\theta$

$\Rightarrow dx = 2\cos\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ θ varies from 0 to $\frac{\pi}{2}$

Substituting the corresponding values in equation (3),

$$V = \frac{8}{3}\int_0^{\frac{\pi}{2}} (4-(2\sin\theta)^2)^{\frac{3}{2}} 2\cos\theta d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} (4-4\sin^2\theta)^{\frac{3}{2}} \cos\theta\, d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} 4(1-\sin^2\theta)^{\frac{3}{2}} \cos\theta\, d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} (4\cos^2\theta)^{\frac{3}{2}} \cos\theta\, d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} (2^2)^{\frac{3}{2}} (\cos^2\theta)\cos\theta\, d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{2}} 8.\cos^3\theta\cos\theta\, d\theta$$

$$= \frac{128}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= \frac{128}{3}.\frac{4-1}{4}.\frac{4-3}{4-2}.\frac{\pi}{2}$$

$$\left[\because \int_0^{\frac{\pi}{2}} \cos^n\theta d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}....\frac{1}{2}\frac{\pi}{2}\right]$$

$$= \frac{128}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$= 8\pi$ cubic units

$\therefore$ Volume of the region is 8π cubic units.

Q66. Find by using triple integrals, the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x + y + z = a.

OR

Find the volume the tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Answer :

Given planes are,

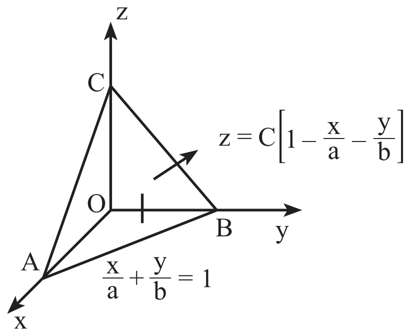
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad ... (1)$$

$x = 0$

$y = 0$

The region of integration is bounded by the planes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, $x = 0$, $y = 0$ and $z = 0$.

Figure represents the plane *OABC*.



Figure

From equation (1)

$$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b}$$

$$\Rightarrow \quad z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

$\therefore$ z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

Consider *xy* plane.

Substituting $z = 0$ in equation (1),

$$\frac{x}{a} + \frac{y}{b} = 1 \quad ... (2)$$

$$\Rightarrow \quad \frac{y}{b} = 1 - \frac{x}{a}$$

$$\Rightarrow \quad y = b\left(1 - \frac{x}{a}\right)$$

$\therefore$ y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

The projection of tetrahedron in *xy* plane is the triangle *OAB* bounded by $x = 0$, $y = 0$, $\frac{x}{a} + \frac{y}{b} = 1$

Volume of the tetrahedron is given as,

$$V = \int\int_D\int dx\, dy\, dz$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} [z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[c\left(1-\frac{x}{a}-\frac{y}{b}\right) - 0\right] dy\, dx$$

$$= c\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[\left(1-\frac{x}{a}\right) - \frac{y}{b}\right] dy\, dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)(y) - \frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)}$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)\left(b\left(1-\frac{x}{a}\right)\right)-\left[b\left(1-\frac{x}{a}\right)\right]^2\right]dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{2}\right)^2 - \frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right]dx$$

$$= cb\int_0^a \left[\left(1-\frac{x}{a}\right)^2 - \left(\frac{1-\frac{x}{a}}{2}\right)^2\right]dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^{2+1}}{2+1\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= -\frac{abc}{6}\left[\left(1-\frac{x}{a}\right)^3\right]_0^a$$

$$= -\frac{abc}{6}\left[\left(1-\frac{a}{a}\right)^3 - (1-0)^3\right]$$

$$= -\frac{abc}{6}(0-1)$$

$$= \frac{abc}{6} \text{ cubic units}$$

$\therefore$ Volume of the tetrahedron is $\frac{abc}{6}$ cubic units.

Q67. Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$, where V is the region of space bounded by planes by x = 0, x = 1, y = 0, y = 2 and z = 0, z = 3.

Answer :

Given integral is,

$$\iiint_V (xy+yz+zx)\,dx\,dy\,dz$$

Limits

x varies from 0 to 1

y varies from 0 to 2

z varies from 0 to 3.

$$\therefore \quad I = \int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1} (xy+yz+zx)\,dxdydz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left[\int_{x=0}^{1}(xy+yz+zx)dx\right]dydz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left[y\frac{x^2}{2}+xyz+z\frac{x^2}{2}\right]_0^1 dydz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left(y.\frac{1^2}{2}+(1)yz+z\frac{1^2}{2}\right)dydz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left(\frac{y}{2}+yz+\frac{z}{2}\right)dydz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\frac{y+2yz+z}{2}\,dydz$$

$$\Rightarrow \quad I = \frac{1}{2}\int_{z=0}^{3}\left[\int_{y=0}^{2}(y+2yz+z)dy\right]dz$$

$$= \frac{1}{2}\int_{z=0}^{3}\left[\frac{y^2}{2}+2z\frac{y^2}{2}+yz\right]_0^2 dz$$

$$= \frac{1}{2}\int_{z=0}^{3}\left[\frac{2^2}{2}+z(2)^2+2z\right]dz$$

$$= \frac{1}{2}\int_{z=0}^{3}(2+4z+2z)\,dz$$

$$= \frac{1}{2}\int_{z=0}^{3}(2+6z)\,dz$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\int_{z=0}^{3}(2+6z)dz\right]$$

$$= \frac{1}{2}\left[2z+\frac{6z^2}{2}\right]_0^3$$

$$= \frac{1}{2}\left[2(3)+3(3)^2\right]$$

$$= \frac{1}{2}(6+27) = \frac{33}{2}$$

$$\therefore \iiint_V (xy+yz+zx)dx\,dy\,dz = \frac{33}{2}$$

UNIT 5

VECTOR CALCULUS

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. Define the following,

(i) Gradient

(ii) Divergence

(iii) Curl.

Answer :

(i) Gradient

The gradient of scalar point function f is defined as,

$$\text{grad } f = \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

(ii) Divergence

The divergence of a continuously differentiable vector point function 'F' can be defined as,

$$\text{div } F = \nabla . F = i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$$

Where,

$$F = fi + \phi j + \psi k$$

Then,

$$\text{div } F = \nabla . F = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} . (fi + \phi j + \psi k)$$

$$\nabla . F = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

(iii) Curl

The curl of a continuously differentiable vector point function 'F' can be expressed as,

$$\text{Curl } F = \nabla \times F = i\times\frac{\partial F}{\partial x} + j\times\frac{\partial F}{\partial y} + k\times\frac{\partial F}{\partial z}$$

If,

$$F = fi + \phi j + \psi k$$

Then,

$$\text{Curl } F = \nabla \times F = i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z} \times (fi + \phi j + \psi k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + j\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + k\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right)$$

Q2. List out the properties related to gradient, divergence and curl.

Answer :

The following are the properties related to gradient, divergence and curl.

- div grad $f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$
- curl grad $f = \nabla \times \nabla f = 0$
- div curl $F = \nabla.\nabla \times F = 0$
- curl curl F = grad div $F - \nabla^2 F$

  i.e, $\nabla \times (\nabla \times F) = \nabla (\nabla.F) - \nabla^2 F$
- grad div F = Curl curl $F + \nabla^2 F$

  i.e., $\nabla (\nabla.F) = \nabla \times (\nabla \times F) + \nabla^2 F$
- grad $(fg) = f(\text{grad } g) + g(\text{grad } f)$

  i.e., $\nabla (fg) = f \nabla g + g \nabla f$
- div (fG) = grad $f.G + f(\text{div } G)$

  i.e., $\nabla.(fG) = \nabla f.\ G + f\nabla.G$
- curl $(fG) = (\text{grad } f) \times G + f(\text{curl } G)$

  i.e., $\nabla \times (fG) = \nabla f \times G + f\nabla \times G$
- grad $(F.G) = (F.\nabla)G + (G.\nabla) F + F \times \text{curl } G + G \times \text{curl } F$

  i.e., $(\nabla(F.G) = (F.\nabla) G + (G.\nabla) F + F \times (\nabla \times G) + G \times (\nabla \times F))$
- div $(F \times G) = G.(\text{curl } F) - F.(\text{curl } G)$

  i.e., $(\nabla.F \times G) = G.\ (\nabla \times F) - (\nabla \times G)$
- curl $(F \times G) = F(\text{div } G) - G(\text{div } F) + (G.\nabla) F - (F.\nabla)G$

  i.e., $\nabla \times (F \times G) = F(\nabla.G) - G (\nabla.F) + (G.\nabla) F - (F.\nabla) G$

Q3. Compute the gradient of the scalar function f(x, y, z) = e^{xy} (x + y + z) at (2, 1,1).

Answer : Dec.-16, Q9

Given that,

$f(x, y, z) = e^{xy} (x + y + z)$

Point, $p = (2, 1,1)$

Gradient of a function is given as,

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i\ \frac{\partial}{\partial x}\left[e^{xy}(x+y+z)\right] + j\frac{\partial}{\partial y}\left[(e^{xy}(x+y+z)\right] + k\frac{\partial}{\partial z}[e^{xy}(x+y+z)$$

$$= i\ [e^{xy}y(x+y+z)+e^{xy}] + j[e^{xy}x(x+y+z)+e^{xy}] + ke^{xy}$$

$$\nabla f\,|_{(2,1,1)} = i[e^{2.1}(1)\ (2+1+1) + e^{2(1)}] + j\ [e^{2(1)}\ (2)\ (2+1+1) + e^{2.1}] + ke^{2(1)}$$

$$= i[e^2\,4 + e^2] + j\ [e^2(8) + e^2] + ke^2$$

$$= 5e^2 i + 9e^2 j + ke^2 = e^2\ [5i + 9j + k]$$

$$\therefore\ \nabla f\,|_{(2,1,1)} = e^2\ [5i + 9j + k]$$

Q4. Evaluate ∇^2 log r where r = $\sqrt{x^2+y^2+z^2}$

Answer :

Given that,

$$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$$

But, $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$$\nabla^2 \log r = \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right] \log (x^2+y^2+z^2)^{1/2}$$

$$= \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\frac{1}{2}\log(x^2+y^2+z^2) \qquad [\because n\log a = \log a^n]$$

$$= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\log(x^2+y^2+z^2) + \frac{\partial^2}{\partial y^2}\log(x^2+y^2+z^2) + \frac{\partial^2}{\partial z^2}\log(x^2+y^2+z^2)\right]$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial z}\left[\frac{2z}{x^2+y^2+z^2}\right]\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial z}\left[\frac{z}{x^2+y^2+z^2}\right]$$

$$= \frac{(x^2+y^2+z^2)-x(2x)}{(x^2+y^2+z^2)^2}+\frac{(x^2+y^2+z^2)-y(2y)}{(x^2+y^2+z^2)^2}+\frac{(x^2+y^2+z^2)-z(2z)}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2-2x^2+x^2+y^2+z^2-2y^2+x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{y^2+z^2-x^2+x^2+z^2-y^2+x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{1}{x^2+y^2+z^2}$$

$$= \frac{1}{r^2} = r^{-2} \qquad [\because x^2+y^2+z^2 = r^2]$$

$\therefore \quad \nabla^2 \log r = r^{-2}$

Q5. If $\vec{r}$ is a position vector of the point P(x, y, z) then prove that $\nabla f(r) = f'(r)\frac{\vec{r}}{|\vec{r}|}$.

Answer :

Given that,

$\vec{r}$ is a position vector of the point $P(x, y, z)$.

i.e., $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

Let, $\phi = f(r)$

$$\frac{\partial\phi}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\frac{x}{r} \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

Similarly,

$$\frac{\partial \phi}{\partial y} = f'(r)\frac{y}{r} \text{ and } \frac{\partial \phi}{\partial z} = f'(r)\frac{z}{r}$$

Consider, $\nabla(f(r))$

$$= \nabla(\phi) = \frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k}$$

$$= f'(r)\frac{x}{r}.\bar{i} + f'(r).\frac{y}{r}\bar{j} + f'(r).\left(\frac{z}{r}\right)\bar{k}$$

$$= \frac{f'(r)}{r}[x\bar{i} + y\bar{j} + z\bar{k}]$$

$$= \frac{f'(r)}{r}.\bar{r} \qquad (\because \bar{r} = x\bar{i} + y\bar{j} + z\bar{k})$$

$$\therefore \quad \nabla(f(r)) = f'(r)\frac{\bar{r}}{|\bar{r}|}$$

Q6. In what direction from (3, 1, –2), direction derivative of f = $x^2y^2z^4$ is maximum. Find the Maximum value.

Answer :

Given function is,

$$f = x^2y^2z^4 \qquad \text{... (1)}$$

Point, $P(3, 1, -2)$

Partially differentiating equation (1) with respect to 'x', 'y' and 'z',

$$\frac{\partial f}{\partial x} = 2xy^2z^4$$

$$\frac{\partial f}{\partial y} = 2yx^2z^4$$

$$\frac{\partial f}{\partial z} = 4x^2y^2z^3$$

$$\text{grad } f = \nabla f = \sum \bar{i}\frac{\partial f}{\partial x}$$

$$= \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}(2xy^2z^4) + \bar{j}(2yx^2z^4) + \bar{k}(4x^2y^2z^3)$$

∇f at the point (3, 1, –2) is,

$$\nabla f = \bar{i}\left(2(3)(1)^2(-2)^4\right) + \bar{j}\left(2(1)(3)^2(-2)^4\right) + \bar{k}\left(4(3)^2(1)^2(-2)^3\right)$$

$$\Rightarrow \quad \nabla f = 96\bar{i} + 288\bar{j} - 288\bar{k}$$

$$\Rightarrow \quad \nabla f = 96(\bar{i} + 3\bar{j} - 3\bar{k})$$

$\therefore$ The direction derivative is maximum in the direction, $96(\bar{i} + 3\bar{j} - 3\bar{k})$

$$|\nabla f| = \sqrt{96^2\left(1^2 + 3^2 + (-3)^2\right)}$$

$$= \sqrt{96^2(19)}$$

$$= 96\sqrt{19}$$

$\therefore$ The maximum value of direction derivative is, $96\sqrt{19}$.

Q7. Find grad ϕ where $\phi = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2-y^2-z^2}}$

Answer :

Given that,

$$\phi = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2+y^2+z^2}} \qquad \dots (1)$$

Gradient of a function, ϕ is expressed as,

$$\text{grad } \phi = \nabla\phi = \left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right]\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} \qquad \dots (2)$$

Consider, $\frac{\partial\phi}{\partial x}$,

$$\frac{\partial\phi}{\partial x} = -(x^2+y^2+z^2)\; e^{-\sqrt{x^2+y^2+z^2}} \cdot \frac{(2x)}{2\sqrt{x^2+y^2+z^2}} + e^{-\sqrt{x^2+y^2+z^2}}.2x \qquad \left(\because \frac{\partial}{\partial x}u.v = u\times\frac{\partial v}{\partial x} + v\times\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial\phi}{\partial x} = x.e^{-\sqrt{x^2+y^2+z^2}}\left[-\frac{(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} + 2\right]$$

$$\frac{\partial\phi}{\partial x} = x.e^{-\sqrt{x^2+y^2+z^2}}\left[2 - \frac{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\right]$$

$$\frac{\partial\phi}{\partial x} = x.e^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right]$$

Similarly,

$$\frac{\partial\phi}{\partial y} = y.e^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right]$$

$$\frac{\partial\phi}{\partial z} = z.e^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right]$$

Substituting the corresponding values in equation (2)

$$\nabla\phi = \bar{i}\,xe^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right] + \bar{j}\,ye^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right] + \bar{k}\,ze^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right]$$

$$= e^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right]\left(x\bar{i} + y\bar{j} + z\bar{k}\right)$$

$$\therefore \quad \nabla\phi = e^{-\sqrt{x^2+y^2+z^2}}\left[2-\sqrt{x^2+y^2+z^2}\right](x\bar{i} + y\bar{j} + z\bar{k})$$

Q8. Find the directional derivative of $2xy + z^2$ at (1, – 1, 3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.

Answer :

The given function is,

$$2xy + z^2$$

$$\vec{r} = \bar{i} + 2\bar{j} + 3\bar{k}$$

Let, $\phi(x, y, z) = 2xy + z^2$ ... (1)

The expression for the directional derivative of 'ϕ' in the direction of $\vec{r}$ at (1, – 1, 3) is given as,

Directional derivative at (1, – 1, 3) $= [\nabla\phi]_{at(1,-1,3)} \cdot \hat{r}$ ... (2)

Where,

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{k\partial z} \qquad \dots (3)$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{1+4+9}} \qquad \dots (4)$$

$$\Rightarrow \quad \hat{r} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{14}} \qquad \dots (5)$$

Partially differentiating equation (1) with respect to x, y and z,

$$\frac{\partial \phi}{\partial x} = 2y$$

$$\frac{\partial \phi}{\partial y} = 2x$$

$$\frac{\partial \phi}{\partial z} = 2z$$

Substituting the corresponding values in equation (3),

$$\nabla\phi = \hat{i}2y + \hat{j}2x + \hat{k}2z$$

$$[\nabla\phi]_{at(1,-1,3)} = \bar{i}2(-1) + \bar{j}2(1) + \bar{k}2(3)$$

$$= -2\bar{i} + 2\bar{j} + 6\bar{k} \quad \text{... (6)}$$

Substituting equations (4) and (6) in equation (2),

$$\text{Directional derivation at}(1,-1.3) = (-2\bar{i} + 2\bar{j} + 6\bar{k}).\frac{(\bar{i} + 2\bar{j} + 3\bar{k})}{\sqrt{14}}$$

$$= \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

$\therefore$ Directional derivative at (1, – 1, 3) = $\frac{20}{\sqrt{14}}$

Q9. Find a unit normal vector to the surface $x^2 + y^2 + 2z^2 = 26$ at the point (2, 2, 3).

Answer :

Given that,

$$f = x^2 + y^2 + 2z^2 - 26$$

Partial derivative of f with respect to x is, $\frac{\partial f}{\partial x} = 2x$

Partial derivative of f with respect to y is, $\frac{\partial f}{\partial y} = 2y$

Partial derivative of f with respect z is, $\frac{\partial f}{\partial z} = 4z$

$$\text{grad } f = \nabla f = \sum i\frac{\partial f}{\partial x}$$

$$= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2x) + j(2y) + k(4z)$$

∇f at the point (2, 2, 3) is given as,

$$\nabla f = 2(2)i + 2(2)j + 4(3)k$$

$$\Rightarrow \nabla f = 4i + 4j + 12k$$

Therefore, ∇f is the normal vector to the surface function, $f = x^2 + y^2 + 2z^2 = 26$

The required unit normal vector is given as,

$$\frac{\nabla f}{|\nabla f|} = \frac{4i + 4j + 12k}{\sqrt{(4)^2 + (4)^2 + (12)^2}}$$

$$= \frac{4(i + j + 3k)}{\sqrt{16 + 16 + 144}}$$

$$= \frac{4(i + j + 3k)}{4\sqrt{11}} = \frac{i + j + 3k}{\sqrt{11}}$$

$\therefore$ Unit vector normal to the surface = $\frac{i + j + 3k}{\sqrt{11}}$.

Q10. Find the angle between the two surfaces $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 - z = 3$ at (2, –1, 2).

Answer :

Given surfaces are,

$$x^2 + y^2 + z^2 = 9 \quad \text{... (1)}$$

$$x^2 + y^2 - z = 3 \quad \text{... (2)}$$

Point $P(2, -1, 2)$

Angle between two surfaces of equations (1) and (2) is the angle between the normal to the surfaces at the point (2, –1, 2).

Let, $\phi_1 = x^2 + y^2 + z^2 - 9$ and

$\phi_2 = x^2 + y^2 - z - 3$

The normal to first surface is expressed as,

$$\overline{n_1} = \nabla\phi_1$$

$$= \bar{i}\frac{\partial}{\partial x}\phi_1 + \bar{j}\frac{\partial}{\partial y}\phi_1 + \bar{k}\frac{\partial}{\partial z}\phi_1$$

$$\left[\because \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$$

$\overline{n_1}$ at (2, –1, 2) = $4\bar{i} - 2\bar{j} + 4\bar{k}$

Similarly, normal to second surface is,

$$\overline{n_2} = \nabla\phi_2$$

$$= \bar{i}\frac{\partial}{\partial x}\phi_2 + \bar{j}\frac{\partial}{\partial y}\phi_2 + \bar{k}\frac{\partial}{\partial z}\phi_2$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(-1)$$

$\overline{n_2}$ at (2, –1, 2) = $4\bar{i} - 2\bar{j} - \bar{k}$

Angle between the normal is $\cos\theta = \frac{\bar{n}_1.\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

Substituting the corresponding values in above equation,

$$= \frac{(4\bar{i} - 2\bar{j} + 4\bar{k}).(4\bar{i} - 2\bar{j} - \bar{k})}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}}$$

$$= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \cos\theta = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

$\therefore$ The angle between two surfaces is $\theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$

Q11. If $\overline{f} = xy^2\,\overline{i} + 2x^2yz\,\overline{j} - 3yz^2\,\overline{k}$ then find div $\overline{f}$ at (1, –1, 1).

Answer :

Given,

$$\overline{f} = xy^2\,\overline{i} + 2x^2yz\overline{j} - 3yz^2\overline{k}$$

$$\text{div}\,\overline{f} = \nabla.f = \left(\overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}\right).(xy^2\,\overline{i} + 2x^2yz\,\overline{j} - 3yz^2\overline{k})$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2\frac{\partial}{\partial x}(x) + 2x^2z\,\frac{\partial}{\partial z}(y) - 3y\frac{\partial}{\partial z}(z^2)$$

$$= y^2(1) + 2x^2z(1) - 3y(2z)$$

$\Rightarrow$ div $\overline{f} = 2x^2z + y^2 - 6yz$

div $\overline{f}$ at (1, –1, 1) $= 2(1)^2\,(1) + (-1)^2 - 6(-1)\,(1)$

$= 2 + 1 + 6 = 9$

$\therefore$ div $\overline{f}$ at (1, –1, 1) = 9

Q12. If ω is constant vector, evaluate curl V where V = ω × r̄.

Answer :

Given that,

ω is any constant vector such that,

$$V = \omega \times \overline{r}$$

$$\text{curl}\,V = \text{curl}\,(\omega \times \overline{r})$$

$$= \Sigma\overline{i} \times \left(\frac{\partial}{\partial x}(\omega \times \overline{r})\right)$$

$$= \Sigma\overline{i} \times \left[\frac{\partial\omega}{\partial x} \times \overline{r} + \omega \times \frac{\partial\overline{r}}{\partial x}\right] \qquad \left[\because \frac{\partial}{\partial x}(\overline{a}\times\overline{b}) = \frac{\partial\overline{a}}{\partial x}\times\overline{b} + \overline{a}\times\frac{\partial\overline{b}}{\partial x}\right]$$

$$= \Sigma\overline{i} \times [\overline{0} \qquad \times\overline{i}]$$

$$= \Sigma\overline{i} \times (\quad \times\overline{i}) = \Sigma[(\overline{i}.\overline{i}) \quad - (\overline{i}.\omega)\overline{i}] \qquad [\because \overline{a}\times(\overline{b}\times\overline{c}) = (\overline{a}.\overline{c})\overline{b} - (\overline{a}.\overline{b})\overline{c}]$$

$$= \Sigma\,\omega - \Sigma\,(\overline{i}.\omega)\,i$$

$\therefore$ curl $V = 3\omega - \omega = 2\omega$

Q13. Define laplacian operator.

Answer :

The laplacian operator (∇^2) is defined as,

$$\nabla.\nabla\phi = \left(\overline{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}\right).\left(i\frac{\partial\phi}{\partial x} + \overline{j}\frac{\partial\phi}{\partial y} + \overline{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \overline{i}.\overline{i}\frac{\partial}{\partial x}.\frac{\partial\phi}{\partial x} + \overline{i}.\overline{j}\frac{\partial}{\partial x}.\frac{\partial\phi}{\partial y} + \overline{i}.\overline{k}\frac{\partial}{\partial x}\frac{\partial\phi}{\partial z} + \overline{j}.\overline{i}\frac{\partial}{\partial y}.\frac{\partial\phi}{\partial x} + \overline{j}.\overline{j}\frac{\partial}{\partial y}\frac{\partial\phi}{\partial y} + \overline{j}.\overline{k}\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} + \overline{k}\overline{i}\frac{\partial}{\partial z}\frac{\partial\phi}{\partial x} + \overline{k}.\overline{j}\frac{\partial}{\partial z}.\frac{\partial\phi}{\partial y} + \overline{k}.\overline{k}\frac{\partial}{\partial z}\frac{\partial\phi}{\partial z}$$

$$\left[\begin{array}{l} \because \overline{i}.\overline{i} = \overline{k}.\overline{k} = \overline{j}.\overline{j} = 1 \\ \overline{i}.\overline{j} = \overline{j}.\overline{k} = \overline{i}.\overline{k} = \overline{k}.\overline{j} \\ \overline{j}.\overline{i} = \overline{k}.\overline{i} = 0 \end{array}\right]$$

$$= \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$\therefore$ $\nabla.\nabla\phi = \nabla^2\phi$

Q14. If F is a conservative vector field show that curl F = 0.

Answer :

For a conservative vector field, (F)

$$F = \nabla \phi$$

$$F = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

Then,

$$\text{Curl } F = \nabla \times F$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right) - \hat{j}\left(\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial x}\right)\right) + \hat{k}\left(\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right)$$

$$= \hat{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

$\therefore \nabla \times F = 0$

Q15. If A is a vector function, find div(Curl A).

Answer : Aug.-14, Set-2, Q1(vii)

The given vector function $\vec{A}$ can be written as,

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\text{Curl } \vec{A} = \Delta \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\text{div (Curl } A) = \nabla.(\nabla \times A) = \left(\frac{\hat{i}\partial}{\partial x} + \frac{\hat{j}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right) \left(\hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_1}{\partial y \partial z} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}$$

$$= 0$$

$\therefore$ div (Curl) $A = 0$

Q16. If f is a differentiable scalar field, then show that $\nabla \times \nabla f = \vec{0}$.

Answer : **Dec.-16, Q10**

Given that,

f is a differentiable scalar field.

Consider,

$\nabla \times (\nabla f)$ = Curl (Grad f)

$$\nabla f = \left[i\frac{\partial}{\partial x}f + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right]$$

$$= \left[\frac{\partial f}{\partial x}\right]i + \left[\frac{\partial f}{\partial y}\right]j + \left[\frac{\partial f}{\partial z}\right]k$$

$$\Rightarrow \quad \text{Curl (Gradf)} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right] - j\left[\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right] + k\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right]$$

$= 0$

$\therefore \quad \nabla \times (\nabla f) = \vec{0}$

Q17. Show that the vector $e^{x+y-2z}(\hat{i} + \hat{j} + \hat{k})$ is solenoidal.

Answer : **June/July-17, Q10**

Given function is,

$$f = e^{x+y-2z}(\hat{i} + \hat{j} + \hat{k})$$

A function $\hat{F}$ is said to be solenoidal if it satisfies the condition,

$$\nabla . f = 0$$

Since, $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$$\Rightarrow \quad \nabla . f = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right] . e^{x+y-2z}(i + j + k)$$

$$= \frac{\partial}{\partial x}(e^{x+y-2z}) + \frac{\partial}{\partial y}(e^{x+y-2z}) + \frac{\partial}{\partial z}(e^{x+y-2z})$$

$$= e^{x+y-2z}(1) + e^{x+y-2z}(1) + e^{x+y-2z}(-2)$$

$$= e^{x+y-2z} + e^{x+y-2z} - 2e^{x+y-2z}$$

$$= 2^{ex+y-2z} - 2e^{x+y-2z}$$

$$= 0$$

$\Rightarrow \quad \nabla . f = 0$

$\therefore$ The vector is solenoidal.

Q18. If ϕ satisfies Laplace equation, show that $\nabla\varphi$ is both solenoidal and irrotational.

Answer :

Given scalar point function is, ϕ

'ϕ' satisfies Laplace equation

i.e., $\nabla^2\phi = 0$

$\Rightarrow \quad \nabla.(\nabla\phi) = 0$ ... (1)

A vector point function $\vec{f}$ is said to be solenoidal if and only if,

$\nabla . \vec{f} = 0$ ... (2)

Comparing equations (1) and (2),

$\nabla\phi$ is solenoidal

From the property of curl,

$\nabla \times \nabla\phi = 0$ ... (3)

A vector $\vec{f}$ is said to be irrotational if and only if,

$\nabla \times \vec{f} = 0$ ... (4)

Comparing equations (3) and (4),

$\nabla\phi$ is irrotational

$\therefore \quad \nabla\phi$ is both solenoidol and irrotational.

Hence proved.

Q19. Define line integral.

Answer :

Line Integral

Consider a smooth curve 'C' in space, which is defined by function $\bar{r} = \bar{f}(t)$, as shown in figure (1),

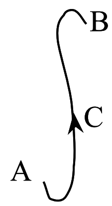


Figure (1)

Where, A and B are the starting and terminating points of a curve respectively.

Differentiating a part of the $\bar{r}$ with respect to 'S' gives unit vector along the tangent to the curve 'C' at point 'P'.

i.e., $\frac{d\bar{r}}{ds} = \bar{t}$

Where,

ds – Differential of arc length at $P \in C$.

Now, consider continuous vector point function $\overline{F}(r)$ which is defined along curve 'C'. The component of $\overline{F}(r)$ along the tangent at point 'P' is given as,

$\overline{F}(r).\bar{t}$

Then, the line integral of $\overline{F}$ along C is given as,

$$\int_C \left(\overline{F}.\frac{\overline{dr}}{ds}\right)ds = \int_C \overline{F}.\overline{dr} = \int_A^B \overline{F}.\bar{t}\ ds$$

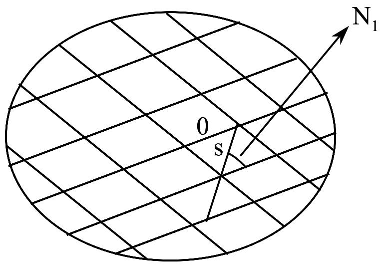
Q20. Define surface integral.

Answer :

Surface Integral

Surface integral is defined as an integral which is evaluated over a surface.

Let's consider a smooth surface $\bar{r} = \bar{f}(u,v)$ and continuous vector point function $\overline{F}(\bar{r})$ which is defined over smooth surface as shown in figure (2).



Figure

Figure (2) represents the region of the surface, which is divided into m sub regions of areas $\delta s_1, \delta s_2 \ldots \delta s_i, \ldots \delta s_m$.

The vector area of S_i is given as,

$$\delta\overline{A}i = \bar{n}_i \delta S_i$$

Where,

$\bar{n}_i$ – Unit normal of δS_i

Consider the sum $I_m = \sum_{i=1}^{m} \overline{F}(\bar{r}_i).\delta A_i = \Sigma\overline{F}(\bar{r}_i).\bar{n}_i.ds_i$

As the number of sub surfaces increases the area of sub surfaces decreases.

i.e., when $m \to \infty$ $\delta s_i \to 0$, if it exists then integral is called normal surface integral which is given as,

$$\int_s \overline{F}(\bar{r}).d\overline{A} \text{ (or) } \int_s \overline{F}.\bar{n}ds$$

Q21. Define volume integral.

Answer :

Volume Integral

Consider a vector point function $\overline{F}(\bar{r})$. If 'V' is the volume enclosing the surface $\bar{r} = \bar{f}(u,v)$, then, divide volume into m sub-regions, $\delta V_1, \delta V_2 \ldots \delta V_i, \ldots \delta V_m$.

If $P_i(\bar{r}_i)$ represents a point in δv_i then, sum is given as,

$$I_m = \sum_{i=1}^{m} \overline{F}(r_i)\delta V_i$$

When, $m \to \infty$, $\delta V_i \to 0$ then volume integral is defined as,

$$\int_V \overline{F}(\bar{r})dv \text{ (or) } \int_V \overline{F}dv$$

Q22. Define work.

Answer :

Work is defined as the integral of force acting on the perpendicular displacement of a particle over some distance.

Let F be the force acting over a particle moving along an arc AB, then the work done during a small displacement SR is given as,

$$W = \int_A^B F.dR$$

Q23. Define the term potential function.

Answer :

Potential Functions

A vector field that can be easily obtained from a scalar field is defined by a function called vector function. This function is represented by $\overline{F}$, which is equal to the gradient of a scalar field. So that,

$\overline{F}$ = Grad (ϕ)

i.e., $\overline{F} = \nabla\phi$

Where,

$\overline{F}$ = Conservative vector and

ϕ = Scalar potential.

Note

If $\overline{F}$ is conservative or irrotational then $\overline{F} = \nabla\phi$

Q24. Prove that the work done by a force F depends on the end points and not on the path in a conservative field.

Answer :

Let $\overline{F}$ represent the force acting on a particle moving from A to B. During a small displacement δr, the work done is $\overline{F}.\delta r$.

Total work done from A to B $= \int_A^B \overline{F}\, d\bar{r}$

As the force $\overline{F}$ is conservative, there exists a scalar function 'g' such that,

$$\overline{F} = \nabla g$$

$$= \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$$

$\therefore$ Work done,

$$= \int_A^B \overline{F}.d\bar{r} \int_A^B \left[\bar{i}\frac{\partial \bar{g}}{\partial x} + \bar{j}\frac{\partial \bar{g}}{\partial y} + \bar{k}\frac{\partial \bar{g}}{\partial z}\right] \times \left[\bar{i}dx + \bar{j}dy + \bar{k}dz\right]$$

$$= \int_A^B \left[\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right]$$

$$= \int_A^B dg = [g]_A^B = B - A$$

Thus, in a conservative field, the work done depends on the end points A and B and not on the path from A to B.

Q25. Find the work done by the force $\vec{F}$ = 3x²i + (2xz – y)j + zk along the straight line joining the points (0, 0, 1) and (2, 1, 3).

Answer :

Given that,

$$\vec{F} = 3x^2 i + (2xz - y)j + zk$$

Points are (0, 0, 1) to (2, 1, 3)

Work done $(W) = \int_C \vec{F}.d\vec{r}$

Where,

$$d\vec{r} = dxi + dyj + dzk$$

$$\Rightarrow \quad \vec{F}.d\vec{r} = 3x^2 dx + (2xz - y)dy + zdz$$

$$\Rightarrow \quad \text{Work done} = \int_{(0,0,1)}^{(2,1,3)} \left[3x^2 dx + (2xz - y)dy + zdz\right]$$

$$= \left[\frac{3x^3}{3} + \left(2xzy - \frac{y^2}{2}\right) + \frac{z^2}{2}\right]_{(0,0,1)}^{(2,1,3)}$$

$$= \left[8 + 2(2)(3) - \frac{1}{2} + \frac{9}{2}\right] - \left[\frac{1}{2}\right]$$

$$= 8 + 12 + 4 - \frac{1}{2}$$

$$= 24 - \frac{1}{2}$$

$$= \frac{47}{2}$$

$$\therefore \quad W = \frac{47}{2} \text{ units.}$$

Q26. If $\overline{F}$ = (5xy – 6x²) $\overline{i}$ + (2y – 4x) $\overline{j}$ then evaluate $\int \overline{F}.\overline{dR}$ along the curve y = x³ from the point (1, 1) to (2, 8).

Answer :

Given that,

$$\overline{F} = (5xy - 6x^2)i + (2y - 4x)j$$

Curve c is, $y = x^3$ ... (1)

(x, y) varies from (1, 1) to (2, 8)

The vector $\vec{r}$ can be written as,

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\Rightarrow dr = dx\hat{i} + dy\hat{j}$$

$$\overline{F}\,.\,\overline{dr} = \left((5xy - 6x^2)i + (2y - 4x)j\right).(dx\hat{i} + dy\hat{j})$$

$$= (5xy - 6x^2)dx + (2y - 4x)dy$$

From equation (1),

$$dy = 3x^2 dx.$$

$$\int_c \overline{F}.\overline{dr} = \int_1^2 \left((5x.x^3 - 6x^2)dx + (2x^3 - 4x)3x^2 dx\right)$$

$$= \int_1^2 \left((5x^4 - 6x^2)dx + (6x^5 - 12x^3)dx\right)$$

$$= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx$$

$$= \left[\frac{5.x^5}{5} - \frac{6.x^3}{3} + \frac{6x^6}{6} - \frac{12.x^4}{4}\right]_1^2$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$

$$= \left[2^5 - 2(2)^3 + (2)^6 - 3(2)^4\right] - \left[(1)^5 - 2(1)^3 + (1)^6 - 3(1)^4\right]$$

$$= 32 - 16 + 64 - 48 - 1 + 2 - 1 + 3$$

$$= 35$$

$$\therefore \int_c \overline{F}.\overline{dr} = 35.$$

Q27. Write the statement of Green's theorem.

Answer :

If 'S' represents a closed region in xy plane bounded by a simple closed curve 'C' and if M, N are continuous functions of x and y, then,

$$\int_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Q28. State Stoke's theorem.

Answer :

Let 'S' be a surface bounded by a closed non-intersecting curve C. If $\vec{F}$ is any differentiable vector point function, then

$$\oint \vec{F}.d\vec{r} = \int \text{curl}\ \vec{F}.\vec{N}\ dS$$

Where 'C' is traversed in positive direction and $\vec{N}$ is outward drawn unit normal vector.

Q29. State Gauss's divergence theorem.

Answer :

Gauss's divergence theorem states that, for a continuously differentiable vector function located in the region E bounded by the closed surface S, then,

$$\int_S F.NdS = \int_E div\ F\ dV$$

Where,

N – Unit external normal vector.

In cartesian form,

For $F(R) = f(x, y, z)\ I + \phi(x, y, z)\ J + \psi(x, y, z)K$ then Gauss's divergence formula is given as,

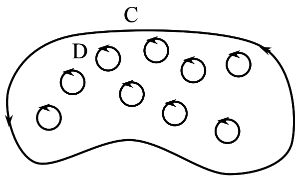
$$\iint_S (fdydz + \phi dzdx + \psi dxdy) = \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dxdydz$$

Q30. Write down the physical interpretation of Green's theorem.

Answer :

Physical Interpretation of Green's Theorem

Green's theorem states that the amount of circulation around '*C*' is equal to the total amount of circulation of all the area inside '*C*'. (i.e., circulation in *D*).



Figure

It can also be said that Green's theorem converts a line integral around a closed curve into a double integral.

$$\therefore \int_C \text{F.ds} = \iint_D \text{circulation of F.dA}$$

Q31. Write down the physical interpretation of stoke's theorem.

Answer :

The cummulative rotational tendency of a fluid to spin across the surface '*S*' is equal to the circulation of a fluid around the boundry curve '*C*'. i.e.,

$$\iint_S (\nabla \times V).\hat{n}ds = \oint_C V.Tds$$

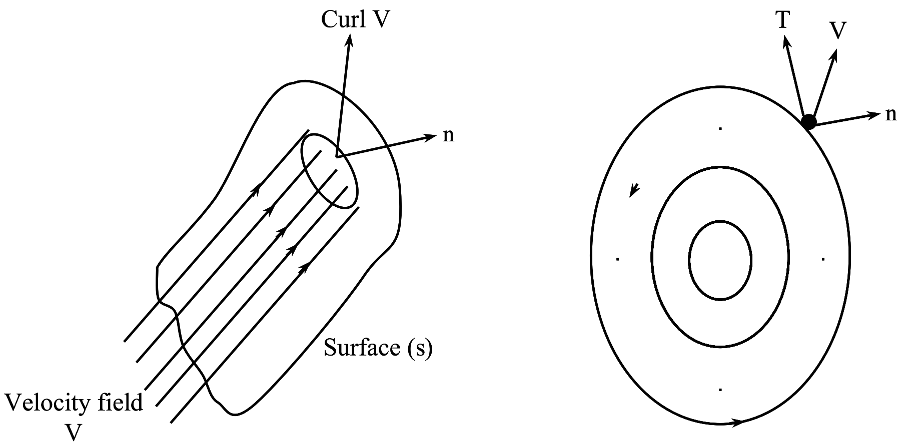
Where,

'*V*' – Velocity field of a fluid flow

Curl '*V*' – Measure of tendency of the fluid to spin or rotate.

$\hat{n}$ – Unit normal to the surface '*S*'.

T – Tangential component of the velocity field '*V*'.



Figure

Q32. Write down the physical interpretation of Gauss's divergence theorem.

Answer :

If a vector field $\vec{F}$ represents the flow of a fluid, then Gauss divergence theorem has the physical interpretation that the total expansion (or contraction) of the fluid inside three dimensional region '*V*' equals the total flux out flow (or inflow) across the boundary of *V*.

i.e., $\iiint_V (\nabla.\vec{F})dv = \iint_{\partial s} \vec{F}.ds$

PART-B
ESSAY QUESTIONS WITH SOLUTIONS

5.1 SCALAR AND VECTOR FIELDS, GRADIENT OF A SCALAR FIELD, DIRECTIONAL DERIVATIVE, DIVERGENCE AND CURL OF A VECTOR FIELD

Q33. Prove that grad $(\vec{F}.\vec{G}) = \vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})+(\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}$.

Answer :

Given equation is,

$$\text{grad }(\vec{F}.\vec{G}) = \vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})+(\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}. \quad ...(1)$$

Consider L.H.S of equation (1),

$$\text{grad }[\vec{F}.\vec{G}] = \nabla[\vec{F}.\vec{G}]$$

$$= i\frac{\partial}{\partial x}[\vec{F}.\vec{G}]+j\frac{\partial}{\partial y}[\vec{F}.\vec{G}]+k\frac{\partial}{\partial z}[\vec{F}.\vec{G}]$$

$$= \sum i\frac{\partial}{\partial x}[\vec{F}.\vec{G}]$$

$$= \left[\sum i\frac{\partial}{\partial x}\vec{F}\right].\vec{G}+\vec{F}.\left[\sum i\frac{\partial}{\partial x}\vec{G}\right] \quad ...(2)$$

Consider,

$$\vec{F}\times[\nabla\times\vec{G}] = \vec{F}\times\left[\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\vec{G}\right]$$

$$= \vec{F}\times\left[\sum i\frac{\partial}{\partial x}\times\vec{G}\right]$$

$$= \vec{F}\times\left[\sum i\times\frac{\partial\vec{G}}{\partial x}\right]$$

$$= \sum\left[\vec{F}.\frac{\partial\vec{G}}{\partial x}\right]i-(\vec{F}.\sum i)\frac{\partial\vec{G}}{\partial x}$$

$$= \sum i\left[\frac{\partial G}{\partial x}.\vec{F}\right]-\vec{F}.\left(\sum i\frac{\partial\vec{G}}{\partial x}\right)$$

$$= \sum i\left[\frac{\partial G}{\partial x}.\vec{F}\right]-(\vec{F}.\nabla)\vec{G}$$

$$\vec{F}\times[\nabla\times\vec{G}]+(\vec{F}.\nabla)\vec{G} = \Sigma i\left(\frac{\partial\vec{G}}{\partial x}.\vec{F}\right) \quad ...(3)$$

Similarly

$$\vec{G}\times[\nabla\times\vec{F}]+(\vec{G}.\nabla)\vec{F} = \Sigma i\left(\frac{\partial F}{\partial x}\right)G \quad ...(4)$$

Substituting equations (3) and (4) in equation (2),

$$\text{grad}[\vec{F}.\vec{G}] = \vec{G}\times[\nabla\times\vec{F}]+(\vec{G}.\nabla)\vec{F}+\vec{F}\times[\nabla\times\vec{G}]+(\vec{F}.\nabla)\vec{G}$$

$$\therefore \quad \text{grad}[\vec{F}.\vec{G}] = \vec{F}\times[\nabla\times\vec{G}]+\vec{G}\times(\nabla\times\vec{F})+(\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}$$

Q34. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then show that, $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $\vec{r} = |\bar{r}|$.

Answer :

Given that,

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$|\bar{r}| = |x\bar{i} + y\bar{j} + z\bar{k}|$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow \quad r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

$$= \sum \frac{\partial^2}{\partial x^2} r^n \qquad \text{... (1)}$$

$$\frac{\partial}{\partial x}(r^n) = n.r^{n-1} \frac{\partial r}{\partial x}$$

$$= n\, r^{n-1} \frac{x}{r} \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= n.x.r^{n-2}$$

$$\frac{\partial^2}{\partial x^2}(r^n) = n\left[r^{n-2}.1 + x(n-2)r^{n-3}\frac{\partial r}{\partial x}\right] = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x^2}{r}\right] = n.r^{n-2} + n(n-2)r^{n-4}x^2$$

Consider equation (1),

$$\nabla^2 r^n = \sum \frac{\partial^2}{\partial x^2} r^n$$

$$= \sum n.r^{n-2} + n(n-2)r^{n-4}x^2$$

$$= 3n\, r^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2)$$

$$= 3n\, r^{n-2} + n(n-2)r^{n-4}(r^2) \qquad [\because r^2 = x^2 + y^2 + z^2]$$

$$= 3n\, r^{n-2} + (n^2 - 2n)\, r^{n-2}$$

$$= r^{n-2}[3n + n^2 - 2n]$$

$$= r^{n-2}[n^2 + n]$$

$$= n(n+1)\, r^{n-2}$$

$$\therefore \quad \nabla^2(r^n) = n(n+1)\, r^{n-2}.$$

Q35. Evaluate ∇^2 log r where r = $x^2 + y^2 + z^2$

Answer :

Given that,

$$r = x^2 + y^2 + z^2$$

But, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\nabla^2 \log r = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \log(x^2 + y^2 + z^2)$$

$$= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \log(x^2 + y^2 + z^2) \qquad [\because n \log a = \log a^n]$$

$$= \left[\frac{\partial^2}{\partial x^2} \log (x^2+y^2+z^2) + \frac{\partial^2}{\partial y^2} \log (x^2+y^2+z^2) + \frac{\partial^2}{\partial z^2} \log (x^2+y^2+z^2) \right]$$

$$= \left[\frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2+z^2}\right] + \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2+z^2}\right] + \frac{\partial}{\partial z}\left[\frac{2z}{x^2+y^2+z^2}\right]\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2+z^2}\right] + \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2+z^2}\right] + \frac{\partial}{\partial z}\left[\frac{2z}{x^2+y^2+z^2}\right]$$

$$= 2\left[\frac{(x^2+y^2+z^2)-x(2x)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)-y(2y)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)-z(2z)}{(x^2+y^2+z^2)^2}\right]$$

$$= 2\left[\frac{x^2+y^2+z^2-2x^2+x^2+y^2+z^2-2y^2+x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}\right]$$

$$= 2.\frac{y^2+z^2-x^2+x^2+z^2-y^2+x^2+y^2-z^2}{(x^2+y^2+z^2)^2} = 2.\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = 2.\frac{1}{x^2+y^2+z^2}$$

$$= \frac{2}{r^2} \qquad [\because x^2+y^2+z^2 = r^2]$$

$$\therefore \quad \nabla^2 \log r = \frac{2}{r^2}$$

Q36. Find the scalar potential (ϕ) such that $\vec{F} = \nabla\phi$ where $\vec{F} = 2xyz^3\,\vec{i} + x^2z^3\,\vec{j} + 3x^2yz^2\,\vec{k}$.

Answer :

Given that,

$$\vec{F} = 2xyz^3\,\vec{i} + x^2z^3\,\vec{j} + 3x^2yz^2\,\vec{k}$$

Gradient of a function is expressed as,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}(\vec{k})$$

$$\vec{F} = \nabla\phi$$

$$\Rightarrow \quad 2xyz^3\,\vec{i} + x^2z^3\,\vec{j} + 3x^2yz^2\,\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

Comparing $\vec{i}$, $\vec{j}$, $\vec{k}$ terms on both sides,

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \qquad \text{... (1)}$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \qquad \text{... (2)}$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 \qquad \text{... (3)}$$

Integrating equations (1), (2) and (3),

$$\int\frac{\partial\phi}{\partial x} = \int 2xyz^3.dx$$

$$\Rightarrow \quad \phi = \frac{2x^2}{2}\,yz^3 + f(y, z) = x^2\,yz^3 + f(y, z) \qquad \text{... (4)}$$

$$\int\frac{\partial\phi}{\partial y} = \int x^2z^3.dy$$

$\Rightarrow \quad \phi = x^2yz^3 + g(z, x)$ ... (5)

$$\frac{\partial\phi}{\partial z} = \int 3x^2yz^2\,dz$$

$$\phi = \frac{3x^2yz^3}{3} + h(x,y)$$

$\Rightarrow \quad \phi = x^2yz^3 + h(x, y)$ ... (6)

The equations (4), (5), (6) will be consistent only when $f(y, z) = g(z, x) = h(x, y) = 0$

$\therefore \quad \phi = x^2yz^3$ + Constant

Q37. Find the directional derivative of the scalar point function ϕ(x, y, z) = 4xy^2 + 2x^{2}yz at the point A(1, 2, 3) in the direction of the line AB where B = (5, 0, 4).

Answer :

Given function is, $\phi(x, y, z) = 4xy^2 + 2x^2yz$.

Point $A = (1, 2, 3)$, Point $B = (5, 0, 4)$

The position vectors of 'A' and 'B' with respect to origin 'O' are,

$\overline{OA} = i + 2j + 3k$ and

$\overline{OB} = 5i + 0j + 4k = 5i + 4k$

$\therefore \overline{AB} = \overline{OB} - \overline{OA} = 5i + 4k - i - 2j - 3k$

$= 4i - 2j + k$ ... (1)

The directional derivative $= e.\nabla\phi$

Where,

'e' = Unit vector in the direction of $\overline{AB}$

$$\Rightarrow \quad e = \left[\frac{4i - 2j + k}{\sqrt{(4)^2 + (-2)^2 + (1)^2}}\right] = \frac{4i - 2j + k}{\sqrt{16+4+1}} = \frac{4i - 2j + k}{\sqrt{21}}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= (4y^2 + 4xyz)i + (8xy + 2x^2z)j + (2x^2y)k$$

Substituting values of 'e' and $\nabla\phi$ in equation (1),

$\therefore$ The directional derivative is,

$$e.\nabla\phi = \frac{4i - 2j + k}{\sqrt{21}}.(4y^2 + 4xyz)i + (8xy + 2x^2z)j + (2x^2y)k$$

$$= \frac{4(4y^2 + 4xyz) - 2(8xy + 2x^2z) + 2x^2y}{\sqrt{21}} \qquad [\because i.i = j.j = k.k = 1]$$

At point (1, 2, 3) $\qquad [\because i.j = j.k = k.i = 0]$

$$e.\nabla\phi = \frac{4(4(2)^2 + 4(1)(2)(3)) - 2(8(1)(2) + 2(1)^2(3)) + 2(1)^2 2}{\sqrt{21}}$$

$$= \frac{4(16+24) - 2(16+6) + 4}{\sqrt{21}} = \frac{160 - 44 + 4}{\sqrt{21}}$$

$$= \frac{160 - 40}{\sqrt{21}} = \frac{120}{\sqrt{21}}$$

$\therefore$ Directional derivative at $A(1,2,3)$ is $\dfrac{120}{\sqrt{21}}$

Q38. Find the directional derivative of f (x, y, z) = $x^2 + y^2 + z^2$ at (1, 2, 3) in the direction of the vector $2\hat{i} + 3\hat{j} + 6\hat{k}$.

Answer : Dec.-17, Q17(b)

Given that,

$f(x, y, z) = x^2 + y^2 + z^2$

Point, $P(x, y, z) = (1, 2, 3)$

Direction vector, $\bar{b} = 2i + 3j + 6k$

The gradient of a scalar function $f(x, y, z)$ is,

$$\nabla f(x, y, z) = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x^2+y^2+z^2) + j\frac{\partial}{\partial y}(x^2+y^2+z^2) + k\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= i(2x) + j(2y) + k\,(2z)$$

$$\nabla f = 2xi + 2jy + 2zk$$

$$\nabla f\ (1, 2,3) = 2(1)\, i + 2(2)\, j + 2(3)\, k$$

$$= 2i + 4j + 6k$$

$$\text{Unit vector, } \hat{b} = \frac{2i + 3j + 6k}{\sqrt{4+9+36}}$$

$$= \frac{2i + 3j + 6k}{\sqrt{49}}$$

$$= \frac{2i + 3j + 6k}{7}$$

The directional derivative of f in the direction of $2i + 3j + 6k$,

$$= \nabla f.\vec{b}$$

$$= (2i + 4j + 6k).\frac{2i + 3j + 6k}{7}$$

$$= \frac{4+12+36}{7} = \frac{52}{7}$$

$\therefore$ Directional derivative $= \frac{52}{7}$.

Q39. Find the directional derivative of $xyz^2 + xz$ at (1, 1, 1) in a direction of the normal to the surface $3xy^2 + y = z$ at (0, 1, 1).

Answer :

Given that,

$f = xyz^2 + xz$

$P_1 = (1, 1, 1)$

$g = 3xy^2 + y - z$

$P_2 = (0, 1, 1)$

Partial derivative of f with respect to x is,

$$\frac{\partial f}{\partial x} = yz^2 + z$$

Partial derivative of f with respect to y is, $\frac{\partial f}{\partial y} = xz^2$

Partial derivative of f with respect to z is, $\frac{\partial f}{\partial z} = 2xyz + x$

$$\text{Grad } f = \nabla f = \sum i\frac{\partial f}{\partial x}$$

$$= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(yz^2 + z) + j\,(xz^2) + k(2xyz + x)$$

$$\therefore \nabla f = i(yz^2 + z) + j\,(xz^2) + k(2xyz + x)$$

$$\nabla f \text{ at } (1, 1, 1) = i[1(1)^2 + 1] + j(1(1)^2) + k(2(1)(1)(1) + 1)$$

$$= 2i + j + 3k$$

Similarly,

$$\frac{\partial g}{\partial x} = 3y^2,\ \frac{\partial g}{\partial y} = 6xy + 1 \text{ and } \frac{\partial g}{\partial z} = -1$$

$$\text{grad g} = \nabla g = i(3y^2) + j(6xy+1) + k(-1)$$

$$\therefore \quad \nabla g = 3y^2 i + (6xy+1)j - k$$

$$\nabla g \text{ at } (0, 1, 1) = 3(1)^2 i + (6(0)\,(1) + 1)j - k$$

$$= 3i + j - k$$

Unit vector normal to surface g is,

$$\frac{\nabla g}{|\nabla g|} = \frac{3i + j - k}{\sqrt{(3)^2 + (1)^2 + (-1)^2}}$$

$$= \frac{3i + j - k}{\sqrt{11}}$$

$$\therefore \quad \frac{\nabla g}{|\nabla g|} = \frac{3i + j - k}{\sqrt{11}}$$

Directional derivative of f in the direction of the normal to the surface g,

$$\text{Directional derivative} = \nabla f\,.\frac{\nabla g}{|\nabla g|}$$

Substituting the corresponding values in above equation,

$$\text{Directional Derivative} = (2i + j + 3k).\left[\frac{3i + j - k}{\sqrt{11}}\right]$$

$$= \frac{(2i)\,(3i) + j.j + (3k)\,(-k)}{\sqrt{11}}$$

$$= \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

$\therefore$ Directional derivative $= \frac{4}{\sqrt{11}}$.

Q40. Find the normal vector and unit normal vector to the surface $z^2 = x^2 - y^2$ at $(2, 1, \sqrt{3})$.

Answer : June/July-17, Q9

Given surface is,

$f = x^2 - y^2 = z^2$

$\Rightarrow \quad f = x^2 - y^2 - z^2$

Point, $P = (2, 1, \sqrt{3})$

Normal vector to the surface is given as,

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \quad ... (1)$$

Substituting the corresponding values in equation (1),

$$\nabla f = i(2x) + j(-2y) + k(-2z)$$

$$= 2xi - 2yj - 2zk$$

$\therefore$ Normal vector = $2xi - 2yj - 2zk$

$$\nabla f|_{(2,1,\sqrt{3})} = 2(2)i - 2(1)\,j - 2(\sqrt{3})k$$

$$= 4i - 2j - 2\sqrt{3}\,k$$

$$|\nabla f|_{(2,1,\sqrt{3})} = \sqrt{(4)^2 + 2^2 + (2\sqrt{3})^2}$$

$$= \sqrt{16 + 4 + 12}$$

$$= \sqrt{32}$$

$$= 2\sqrt{8}$$

Unit normal vector to the surface is given as,

$$\frac{\nabla f}{|\nabla f|} = \frac{\nabla f}{|\nabla f|}\Bigg|_{(2,1,\sqrt{3})} \quad ... (2)$$

Substituting the corresponding values in equation (2),

$$\frac{\nabla f}{|\nabla f|} = \frac{4i - 2j - 2\sqrt{3}\,k}{2\sqrt{8}}$$

$$= \frac{2i - j - \sqrt{3}\,k}{\sqrt{8}}$$

$\therefore$ Unit normal vector = $\dfrac{2i - j - \sqrt{3}\,k}{\sqrt{8}}$.

Q41. Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ cut orthogonally at (1, –1,2).

Answer :

Given surfaces are,

$$f(x, y, z) \Rightarrow 5x^2 - 2yz - 9x = 0 \quad ... (1)$$

$$g(x, y, z) \Rightarrow ax^2y + bz^3 = 4 \quad ... (2)$$

$$p(x, y, z) = (1, -1, 2)$$

$\Rightarrow$ $x = 1, y = -1, z = 2$

Substituting x, y, z values in equation (2)

$$a(1)^2(-1) + b(2)^3 = 4$$

$$\Rightarrow -a + 8b = 4 \quad ... (3)$$

Consider,

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k \quad ... (4)$$

From equation (1),

$$\frac{\partial f}{\partial x} = 5(2x) - 9 = 10x - 9$$

$$\frac{\partial f}{\partial y} = -2z$$

$$\frac{\partial f}{\partial z} = -2y.$$

Substituting the corresponding values in equation (4)

$$\nabla f = (10x - 9)\,i + (-2z)\,j + (-2y)k$$

$$\nabla f|_{(1,-1,2)} = i(10(1) - 9) - 2(2)\,j - 2(-1)k$$

$$\Rightarrow \overline{n_1} = (10 - 9)\,i - 4j + 2k$$

$$\Rightarrow \overline{n_1} = i - 4j + 2k$$

Consider,

$$\nabla g = \frac{\partial g}{\partial x}\,i + \frac{\partial g}{\partial y}\,j + \frac{\partial g}{\partial z}\,k$$

From equation (2)

$$\frac{\partial g}{\partial x} = ay(2x) = 2ayx$$

$$\frac{\partial g}{\partial y} = ax^2$$

$$\frac{\partial g}{\partial z} = b.(3z^2) = 3bz^2$$

Substitutin the corresponding values in equation (5),

$$\nabla g = 2ayxi + ax^2j + 3bz^2k$$

$$\nabla g|_{(1,-1,2)} = 2a(-1)(1)\,i + a(1)^2\,j + 3b(2)^2k$$

$$\Rightarrow \overline{n}_2 = -2ai + aj + 12bk$$

As $\overline{n_1}$ and $\overline{n}_2$ are normal vectors to surfaces (1) and (2) respectively,

$$\overline{n_1}\,.\,\overline{n}_2 = 0$$

$\Rightarrow$ $(i - 4j + 2k).(-2ai + aj + 12bk) = 0$

$\Rightarrow$ $1(-2a) - 4(a) + 2(12b) = 0$

$\Rightarrow$ $-2a - 4a + 24b = 0$

$\Rightarrow$ $-6a + 24b = 0$

$\Rightarrow$ $-6(a - 4b) = 0$

$\Rightarrow$ $a = 4b$.

Substituting $a = 4b$ in equation (3),

$$-4b + 8b = 4$$

$$4b = 4 \Rightarrow b = 1$$

Substituting $b = 1$ in equation (3),

$$8 - 4 = a$$

$\Rightarrow$ $-a + 8(1) = 48$

$\Rightarrow$ $-a + 8 = 4$

$\Rightarrow$ $a = 4$

$\therefore$ $a = 4, b = 1$

Q42. Find the angle between the surfaces $ax^2 + y^2 + z^2 - xy = 1$ and conservative $bx^2y + y^2z + z = 1$ at (1, 1, 0).

Answer :

Given surfaces are,

$ax^2 + y^2 + z^2 - xy = 1$

$bx^2y + y^2z + z = 1$

point $p = (1, 1, 0)$

The angle between the two surfaces is the angle between the normal to the surfaces at (1, 1, 0)

let $\phi_1 = ax^2 + y^2 + z^2 - xy - 1$; $\phi_2 = bx^2y + y^2z + z - 1$

$$n_1 = \nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$n_1 = i(2ax - y) + j(2y - x) + 2zk$$

$$n_1\Big|_{(1,1,0)} = i(2a(1) - 1) + j(2(1) - 1) + 2(0)k$$

$$\Rightarrow \quad n_1\Big|_{(1,1,0)} = (2a - 1)\,i + j$$

$$n_2 = \nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$$

$$= i(2bxy) + j(bx^2 + 2yz) + k(y^2 + 1)$$

$$n_2|_{(1,1,0)} = i(2b(1)(1)) + j(b(1)^2 + 2(1)(0)) + k((1)^2 + 1)$$

$$n_2|_{(1,1,0)} = 2bi + jb + 2k$$

Let θ be the angle between ϕ_1 and ϕ_2, then,

$$\cos\theta = \frac{\overline{n_1}\,.\,\overline{n_2}}{|n_1||n_2|}$$

$$= \frac{((2a-1)i + j).(2bi + jb + 2k)}{\sqrt{(2a-1)^2 + (1)^2}\sqrt{(2b)^2 + (b)^2 + (2)^2}}$$

$$= \frac{(2a-1)(2b) + 1(b)}{\sqrt{(4a^2 - 4a + 2)(4b^2 + b^2 + 4)}}$$

$$= \frac{4ab - 2b + b}{\sqrt{(4a^2 - 4a + 2)(5b^2 + 4)}}$$

$$= \frac{4ab - b}{\sqrt{(4a^2 - 4a + 2)(5b^2 + 4)}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{4ab - b}{\sqrt{(4a^2 - 4a + 2)(5b^2 + 4)}}\right)$$

Q43. If prove that $\nabla.\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$, where $r = \sqrt{x^2 + y^2 + z^2}$.

Answer :

Consider, $\nabla.\left[r\nabla\left(\frac{1}{r^3}\right)\right]$

$$\nabla.\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \nabla.\left[r\nabla\left(\frac{1}{\sqrt{(x^2 + y^2 + z^2)^3}}\right)\right]$$

$$= \nabla.\left[r\nabla\left(\frac{1}{(x^2 + y^2 + z^2)^{3/2}}\right)\right]$$

$$= \nabla\left[r\left[\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right)\right]\right]$$

$$= \nabla\left[r\left[\hat{i}\left(\frac{-3/2\,(x^2+y^2+z^2)^{1/2}(2x)}{(x^2+y^2+z^2)^3}\right)+\hat{j}\left(\frac{-3/2\,(x^2+y^2+z^2)^{1/2}(2y)}{(x^2+y^2+z^2)^3}\right)+\hat{k}\left(\frac{-3/2\,(x^2+y^2+z^2)^{1/2}(2z)}{(x^2+y^2+z^2)^3}\right)\right]\right]$$

$$= \nabla.\left[r\left[\hat{i}\,\frac{-3(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}-\hat{j}\,\frac{3y(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z)^3}-\frac{-3z\hat{k}(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]\right]$$

$$= \nabla.\left[(x^2+y^2+z^2)^{1/2}\,\frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}(-3x\hat{i}-3y\hat{j}-3z\hat{k})\right]$$

$$= \nabla.\left[\frac{1}{(x^2+y^2+z^2)^2}(-3x\hat{i}-3y\hat{j}-3z\hat{k})\right]$$

$$= -3\ \nabla.\frac{x\bar{i}+y\bar{j}+z\bar{k}}{(x^2+y^2+z^2)^2}$$

$$= -3\left[\hat{i}\,\frac{\partial}{\partial x}\left[\frac{x\hat{i}}{(x^2+y^2+z^2)^2}\right]+\hat{j}\,\frac{\partial}{\partial y}\left[\frac{y\hat{j}}{(x^2+y^2+z^2)^2}\right]+\hat{k}\,\frac{\partial}{\partial z}\left[\frac{z\hat{k}}{(x^2+y^2+z^2)^2}\right]\right]$$

$$= -3\left[\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^2}\right]+\frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^2}\right]+\frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^2}\right]\right]$$

$$= -3\left[\frac{(x^2+y^2+z^2)^2-x\times 2\times 2x(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}+\frac{(x^2+y^2+z^2)^2-y\times 2\times 2y(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}+\frac{(x^2+y^2+z^2)^2-z\times 2\times 2z(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}\right]$$

$$= -3\left[\frac{(x^2+y^2+z^2)^2-4x^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}+\frac{(x^2+y^2+z^2)^2-4y^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}+\frac{(x^2+y^2+z^2)^2-4z^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}\right]$$

$$= \frac{-3\,(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}[(x^2+y^2+z^2)-4x^2+x^2+y^2+z^2-4y^2+x^2+y^2+z^2-4z^2]$$

$$= \frac{-3}{(x^2+y^2+z^2)^3}[-3x^2+y^2+z^2-3y^2+x^2+z^2-3z^2+x^2+y^2]$$

$$= \frac{-3}{(x^2+y^2+z^2)^3}[-x^2-y^2-z^2]$$

$$= \frac{3[x^2+y^2+z^2]}{(x^2+y^2+z^2)^3}=\frac{3}{(x^2+y^2+z^2)^2}$$

$$= \frac{3}{(r)^4} = \text{RHS}$$

$$\therefore\ \nabla.\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \frac{3}{(r)^4}$$

Q44. Prove that div.(grad r^m) = m(m + 1) r^{m-2}

Answer :

Given that,

$\nabla^2(r^m) = m(m+1)r^{m-2}$ ... (1)

Position vector, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

And $r = |r|$

$\Rightarrow \quad |\bar{r}| = |x\bar{i} + y\bar{j} + z\bar{k}|$

$\Rightarrow \quad r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$

$\Rightarrow \quad rm = (x^2+y^2+z^2)^{m/2}$

Consider, LHS of equation (1) i.e., $\nabla^2 r^m$

Where,

∇^2 – Laplacian operator

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2(r^m) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^m$$

$$= \sum \frac{\partial^2}{\partial x^2} r^n \quad \text{... (2)}$$

Take $\frac{\partial}{\partial x}(r^m) = m.r^{m-1}\frac{\partial r}{\partial x}$

$$= m\, r^{m-1}\frac{x}{r} \quad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= m.x.r^{m-2}$$

$$\frac{\partial^2}{\partial x^2}(r^m) = m\left[r^{m-2}.1 + x(m-2)r^{m-3}\frac{\partial r}{\partial r}\right]$$

$$= m\left[r^{m-2} + (m-2)r^{m-3}\frac{x^2}{r}\right]$$

$$= m.r^{m-2} + m(m-2)r^{m-4}x^2$$

Consider equation (2), we get,

$$\nabla^2 r^m = \sum \frac{\partial^2}{\partial x^2} r^m$$

$$= \sum m.r^{m-2} + m(m-2)r^{m-4}x^2$$

$$= 3m\, r^{m-2} + m(m-2)r^{m-4}(x^2+y^2+z^2)$$

$$= 3m\, r^{m-2} + m(m-2)r^{m-4}(r^2) \quad [\because r^2 = x^2+y^2+z^2]$$

$$= 3m\, r^{m-2} + (m^2-2m)\, r^{m-2}$$

$$= r^{m-2}[3m + m^2 - 2m]$$

$$= m(m+1)\, r^{m-2}$$

$\therefore \quad \nabla^2(r^m) = m(m+1)\, r^{m-2}$ = R.H.S

Q45. Find curl $\bar{f}$ where $\bar{f}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$.

Answer :

Given that,

$$F = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$$

Let, $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}[x^3 + y^3 + z^3 - 3xyz] + j\frac{\partial}{\partial y}[x^3 + y^3 + z^3 - 3xyz] + k\frac{\partial}{\partial z}[x^3 + y^3 + z^3 - 3xyz]$$

$$= i[3x^2 - 3yz] + j\,[3y^2 - 3xz] + k\,[3z^2 - 3xy]$$

$\therefore \quad F = i[3x^2 - 3yz] + j\,[3y^2 - 3xz] + k\,[3z^2 - 3xy]$

Curl of a function is given as,

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ [3x^2 - 3yz] & [3y^2 - 3xz & [3z^2 - 3xy] \end{vmatrix}$$

$$= i[-3x - (-3x)] - j\,[-3y - (-3y)] + k\,[-3z - (-3z)]$$

$$= 0$$

$\therefore \quad \text{Curl } F = 0.$

Q46. Prove that $\nabla \times \left(\frac{\bar{A} \times \bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$.

Answer :

Given equation is,

$$\nabla \times \left(\frac{\bar{A} \times \bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}} \quad \text{... (1)}$$

But,

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \text{ and } r^2 = x^2 + y^2 + z^2 \quad \text{... (2)}$$

$$\frac{\partial \bar{r}}{\partial x} = \bar{i}, \quad \frac{\partial \bar{r}}{\partial y} = \bar{j}, \quad \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

Differentiating partially equation (1),

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Consider,

$$\text{L.H.S} = \nabla \times \left(\frac{\bar{A}\times\bar{r}}{r^n}\right)$$

$$= \bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right)+\bar{j}\times\frac{\partial}{\partial y}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right)+\bar{k}\times\frac{\partial}{\partial z}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right)$$

$$= \sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \bar{A}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^n}\right)$$

$$= \bar{A}\times\left[\frac{r^n\bar{i}-\bar{r}nr^{n-1}}{r^{2n}}\right]\frac{\partial r}{\partial x}$$

$$= \bar{A}\times\left[\frac{r^n\bar{i}-\bar{r}nr^{n-1}\frac{x}{r}}{r^{2n}}\right] = \bar{A}\times\left[\frac{r^n\bar{i}-\bar{r}nr^{n-2}x}{r^{2n}}\right]$$

$$= \frac{\bar{A}\times i}{r^n}-\frac{n}{r^{n+2}}.x(\bar{A}\times\bar{r})$$

$$\bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \frac{i\times(\bar{A}\times\bar{i})}{r^n}-\frac{nx}{r^{n+2}}.\bar{i}\times(\bar{A}\times\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{A}-(\bar{i}.\bar{A})\bar{i}}{r^n}-\frac{nx}{r^{n+2}}[(\bar{i}.\bar{r})\bar{A}-(\bar{i}.\bar{A})\bar{r}]$$

$$= \frac{\bar{A}-(\bar{i}.\bar{A})\bar{i}}{r^n}-\frac{nx}{r^{n+2}}[(\bar{i}.\bar{r})\bar{A}-(\bar{i}.\bar{A})\bar{r}] \quad \text{... (3)}$$

Suppose,

$$\bar{A} = A_1\bar{i}+A_2\bar{j}+A_3\bar{k} \quad \text{... (4)}$$

Multiplying '$\bar{i}$' on both sides of equation (4),

$$\bar{i}.\bar{A} = A_1$$

Equation (3) becomes,

$$i\times\frac{\partial}{\partial x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \left(\frac{\bar{A}-A_1\bar{i}}{r^n}\right)-\frac{nx}{r^{n+2}}[x\bar{A}-A_1\bar{r}]$$

Taking 'Σ' on both sides,

$$\sum\bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \sum\left(\frac{\bar{A}-A_1\bar{i}}{r^n}\right)-\frac{nx}{r^{n+2}}[x\bar{A}-A_1\bar{r}]$$

Simplifying the above equation,

$$= \frac{3\bar{A}-\bar{A}}{r^n}-\frac{n}{r^{n+2}}(r^2\bar{A})+\frac{n\bar{r}}{r^{n+2}}(A_1x+A_2y+A_3z)$$

$$= \frac{2\bar{A}}{r^n}-\frac{n}{r^n.r^2}(r^2\bar{A})+\frac{n\bar{r}}{r^{n+2}}(\bar{A}\bar{r})$$

$$\therefore\ \nabla\times\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n}+\frac{n\bar{r}}{r^{n+2}}(\bar{A}.\bar{r})$$

Q47. Prove that div (A × B) = B.curl A – A.curl B.

Answer :

Let, $A = A_1 i + A_2 j + A_3 k$ and $\left[\because \begin{array}{l}\text{Curl } A = \nabla\times A\\ \text{Curl } B = \nabla\times B\end{array}\right]$

$B = B_1 i + B_2 j + B_3 k$

Consider,

$$A \times B = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= i\,(A_2B_3 - A_3B_2) - j\,(A_1B_3 - A_3B_1) + k\,(A_1B_2 - A_2B_1)$$

$$= i(A_2B_3 - A_3B_2) + j(A_3B_1 - A_1B_3) + K(A_1B_2 - A_2B_1)$$

$$= \sum i(A_2B_3 - A_3B_2)$$

$$\nabla.(A \times B) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\sum i(A_2B_3 - A_3B_2) \qquad \left[\because \nabla = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\right]$$

$$= \sum \frac{\partial}{\partial x}(A_2B_3 - A_3B_2) \qquad [\because i.i = j.j = k.k = 1]$$

$$= \sum\left(A_2\frac{\partial B_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_3}{\partial x}\right)$$

Consider,

$$(\nabla \times B) = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= i\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) - j\left(\frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z}\right) + k\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$

$$= i\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) + j\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) + k\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$

$$= \sum i\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)$$

$$A.(\nabla \times B) = (A_1 i + A_2 j + A_3 k)\sum i\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)$$

$$\therefore\ A.(\nabla \times B) = \sum A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) \qquad \text{... (1)}$$

Similarly,

$$B.(\nabla \times A) = \sum B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) \qquad \text{... (2)}$$

Subtracting equations (1) and (2),

$$B.(\nabla \times A) - A.(\nabla \times B) = \sum B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \sum A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) \qquad \text{... (3)}$$

Expanding equations (1) and (5),

$$\nabla.(A \times B) = B.(\nabla \times A) - A.(\nabla \times B)$$

(or)

$$\therefore \quad \text{div}(A \times B) = B.\text{curl } A - A.\text{ curl } B$$

Q48. Prove that ∇ × (∇ × A) = –∇²A + ∇(∇.A).

Answer :

Consider,

$$\nabla \times (\nabla \times A) = i \times \frac{\partial}{\partial x}(\nabla \times A) + j \times \frac{\partial}{\partial y}(\nabla \times A) + k \times \frac{\partial}{\partial z}(\nabla \times A) \qquad \left[\because \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]$$

$$\Rightarrow \quad \nabla \times (\nabla \times A) = \sum i \times \frac{\partial}{\partial x}(\nabla \times A)$$

$$i \times \frac{\partial}{\partial x}(\nabla \times A) = i \times \frac{\partial}{\partial x}\left(i \times \frac{\partial A}{\partial x} + j \times \frac{\partial A}{\partial y} + k \times \frac{\partial A}{\partial z}\right)$$

$$= i \times \left(i \times \frac{\partial^2 A}{\partial x^2} + j \times \frac{\partial^2 A}{\partial x \partial y} + k \times \frac{\partial^2 A}{\partial x \partial z}\right)$$

$$= i \times \left(i \times \frac{\partial^2 A}{\partial x^2}\right) + i \times \left(j \times \frac{\partial^2 A}{\partial x \partial y}\right) + i \times \left(k \times \frac{\partial^2 A}{\partial x \partial z}\right)$$

$$= \left(i \frac{\partial^2 A}{\partial x^2}\right) i - \frac{\partial^2 A}{\partial x^2} + \left(i \frac{\partial^2 A}{\partial x \partial y}\right) j + \left(i \frac{\partial^2 A}{\partial x \partial z}\right) k \qquad [\because i.i = 1, i.j = 0, i.k = 0]$$

$$= i\frac{\partial}{\partial x}\left(i\frac{\partial A}{\partial x}\right) + j\frac{\partial}{\partial y}\left(i\frac{\partial A}{\partial x}\right) + k\frac{\partial}{\partial z}\left(i\frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$i \times \frac{\partial}{\partial x}(\nabla \times A) = \nabla\left(i.\frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$\sum i \times \frac{\partial}{\partial x}(\nabla \times A) = \nabla \sum i \frac{\partial A}{\partial x} - \sum \frac{\partial^2 A}{\partial x^2}$$

$$= \nabla(\nabla.A) - \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}\right) = \nabla(\nabla.A) - \nabla^2 A$$

$$\therefore \quad \nabla \times (\nabla \times A) = \nabla(\nabla.A) - \nabla^2 A$$

Q49. Prove that curl(a × b) = a div b – b div a + ($\bar{b}$. ∇) a – (a. ∇)b.

Answer :

Curl (a × b) = ∇ × (a× b) $\qquad \left[\because \nabla.A = i\frac{\partial A}{\partial x} + j\frac{\partial A}{\partial y} + k\frac{\partial A}{\partial z}\right]$

$$= \hat{i} \times \frac{\partial}{\partial x}(\bar{a} \times \bar{b}) + \hat{j} \times \frac{\partial}{\partial y}(\bar{a} \times \bar{b}) + \hat{k} \times \frac{\partial}{\partial z}(\bar{a} \times \bar{b})$$

$$= \hat{i} \times \left[\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x}\right] + \hat{j} \times \left[\frac{\partial \bar{a}}{\partial y} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial y}\right] + \left[\frac{\partial \bar{a}}{\partial z} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial z}\right] \qquad \left(\therefore \frac{\partial}{\partial x}(\bar{a} + \bar{b}) = \frac{\partial \bar{a}}{\partial x} \times (\bar{b} + \bar{a}) \times \frac{\partial \bar{b}}{\partial x}\right)$$

$$= \hat{i} \times \frac{\partial \bar{a}}{\partial x} \times \bar{b} + \hat{i} \times \bar{a}\frac{\partial \bar{b}}{\partial x} \bar{j} \times \frac{\partial \bar{a}}{\partial y} \times \bar{b} + \hat{j} \times \bar{a}\frac{\partial \bar{b}}{\partial y} + \hat{k} \times \frac{\partial \bar{a}}{\partial z} \times \bar{b} + \hat{k} \times \bar{a} \times \frac{\partial \bar{b}}{\partial z}$$

$$= \left[(\hat{i}.\bar{b})\frac{\partial \bar{a}}{\partial x} - \bar{b}\left(\hat{i}\frac{\partial \bar{a}}{\partial x}\right)\right] + \left[\left((\hat{j}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - (\hat{i}.\bar{a})\frac{\partial \bar{b}}{\partial x}\right] + \left[(\hat{j}.\bar{b})\frac{\partial \bar{a}}{\partial y} - \bar{b}\left(\hat{j}.\frac{\partial \bar{a}}{\partial y}\right)\right] + \left[(\hat{j}.\frac{\partial \bar{b}}{\partial y}\bar{a} - (\hat{j}.\bar{a})\frac{\partial \bar{b}}{\partial y}\right]$$

$$+\left((\hat{k}.\bar{b})\frac{\partial \bar{a}}{\partial z} - (\hat{k}\frac{\partial \bar{a}}{\partial z})\bar{b}\right) + \left((\hat{k}.\frac{\partial \bar{b}}{\partial z})\bar{a} - (\hat{k}.\bar{a})\frac{\partial \bar{b}}{\partial z}\right) \qquad \left(\because \bar{a}\times\left(\bar{b}\times\bar{c}\right) = \bar{b}\left(\bar{a}.\bar{c}\right) - c\left(\bar{a}.\bar{b}\right)\right)$$

$$= (\hat{i}.\bar{b})\frac{\partial \bar{a}}{\partial x} + (\hat{j}.\bar{b})\frac{\partial \bar{a}}{\partial y} + (\hat{k}.\bar{b})\frac{\partial \bar{a}}{\partial z} + \left(\hat{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} + \left(\hat{j}\frac{\partial \bar{b}}{\partial y}\right)\bar{a} + \left(\hat{k}.\frac{\partial \bar{b}}{\partial z}\right)\bar{a} - \left(\bar{b}\left(\hat{i}\frac{\partial \bar{a}}{\partial x}\right) + \bar{b}\left(\hat{j}\frac{\partial \bar{a}}{\partial y}\right) + \bar{b}\left(\hat{k}.\frac{\partial \bar{a}}{\partial z}\right)\right)$$

$$-(\hat{i}.\bar{a})\frac{\partial \bar{b}}{\partial x} + (\hat{j}.\bar{a})\frac{\partial \bar{b}}{\partial y} + (\hat{k}.\bar{a})\frac{\partial \bar{b}}{\partial z}$$

$$= (\bar{b}.\nabla)\bar{a} + (\nabla.\bar{b})\bar{a} - (\nabla.\bar{a})\bar{b} - (\bar{a}.\nabla)\bar{b} \qquad \left(\because (\bar{a}.\nabla)f = (\bar{a}.\hat{i})\frac{\partial f}{\partial x} + (\bar{a}.\hat{j})\frac{\partial f}{\partial y} + (\bar{a}.\hat{k})\frac{\partial f}{\partial z}\right)$$

$$= \bar{a}(\nabla.\bar{b}) - \bar{b}(\nabla.\bar{a}) + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= \bar{a}\text{div}\bar{b} - \bar{b}\text{div}\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$\therefore \text{Curl}(a\times b) = \bar{a}\text{div}\bar{b} - \bar{b}\text{div}\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Q50. Show that $\nabla.(\nabla.\bar{F}) = \nabla\times(\nabla\times\bar{F}) + \nabla^2\bar{F}$

Answer :

Consider,

$$\nabla\times(\nabla\times F) = i\times\frac{\partial}{\partial x}(\nabla\times F) + j\times\frac{\partial}{\partial y}(\nabla\times F) + k\times\frac{\partial}{\partial z}(\nabla\times F) \qquad \left[\because \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]$$

$$\Rightarrow \quad \nabla\times(\nabla\times F) = \sum i\times\frac{\partial}{\partial x}(\nabla\times F)$$

$$\Rightarrow \quad i\times\frac{\partial}{\partial x}(\nabla\times F) = i\times\frac{\partial}{\partial x}\left(i\times\frac{\partial F}{\partial x} + j\times\frac{\partial F}{\partial y} + k\times\frac{\partial F}{\partial z}\right)$$

$$= i\times\left(i\times\frac{\partial^2 F}{\partial x^2} + j\times\frac{\partial^2 F}{\partial x\partial y} + k\times\frac{\partial^2 F}{\partial x\partial z}\right)$$

$$= i\times\left(i\times\frac{\partial^2 F}{\partial x^2}\right) + i\times\left(j\times\frac{\partial^2 F}{\partial x\partial y}\right) + i\times\left(k\times\frac{\partial^2 F}{\partial x\partial z}\right)$$

$$= \left(i\frac{\partial^2 F}{\partial x^2}\right)i - \frac{\partial^2 F}{\partial x^2} + \left(i\frac{\partial^2 F}{\partial x\partial y}\right)j + \left(i\frac{\partial^2 F}{\partial x\partial z}\right)k \qquad [\because i.i = 1,\ i.j = 0,\ i.k = 0]$$

$$= i\frac{\partial}{\partial x}\left(i\frac{\partial F}{\partial x}\right) + j\frac{\partial}{\partial y}\left(i\frac{\partial F}{\partial x}\right) + k\frac{\partial}{\partial z}\left(i\frac{\partial F}{\partial x}\right) - \frac{\partial^2 F}{\partial x^2}$$

$$\Rightarrow \quad i\times\frac{\partial}{\partial x}(\nabla\times F) = \nabla\left(i\frac{\partial F}{\partial x}\right) - \frac{\partial^2 F}{\partial x^2}$$

Applying summation on both sides,

$$\Rightarrow \quad \sum i\times\frac{\partial}{\partial x}(\nabla\times F) = \nabla\sum i\frac{\partial F}{\partial x} - \sum\frac{\partial^2 F}{\partial x^2}$$

$$= \nabla(\nabla.F) - \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\right)$$

$$= \nabla(\nabla.F) - \nabla^2 F$$

$$\Rightarrow \quad \nabla\times(\nabla\times F) = \nabla(\nabla.F) - \nabla^2 F$$

$\Rightarrow$ Curl (curl F) = Grad div $\bar{F} - \nabla^2\bar{F}$

$$\therefore \quad \nabla.(\nabla.\bar{F}) = \nabla\times(\nabla\times\bar{F}) + \nabla^2\bar{F}$$

Q51. Find div $\bar{F}$, where $\bar{F} = r^n\ \bar{r}$. Find n if it is solenoidal.

Answer :

Given that,

$\vec{F} = r^n \vec{r}$

$\vec{F}$ is solenoidal,

The vector $\vec{r}$ can be expressed as,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow |r| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$ ... (1)

Partially differentiating equation (1) with respect to x, y and z,

$2r\dfrac{\partial r}{\partial x} = 2x$

$\Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly,

$\Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{y}{r}$

$\Rightarrow \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{div}\,\vec{F} = \nabla.\ r^n \vec{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x\hat{i} + y\hat{j} + z\hat{k})\ r^n$$

$$= \frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n)$$

$$= r^n + nr^{n-1}\frac{\partial r}{\partial x}(x) + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}\,z$$

$$= 3r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right)$$

$$= 3r^n + nr^{n-1}\left(\frac{x.x}{r} + \frac{y.y}{r} + \frac{z.z}{r}\right)$$

$$= 3r^n + nr^{n-1}\left(\frac{x^2 + y^2 + z^2}{r}\right)$$ [∵ From equation (1)]

$$= 3r^n + nr^{n-1}\frac{(r^2)}{r}$$

$$= 3r^n + nr^n$$

$$= r^n(n + 3)$$

$$\therefore div\,\vec{F} = r^n(n+3)$$

The vector $\vec{F}$ is said to be solenoidal if, it satisfies the following condition,

$$\nabla.\vec{F} = 0$$

$$\Rightarrow r^n(n+3) = 0$$

$$\Rightarrow n+3 = 0$$

$$\Rightarrow n = -3$$

$$\therefore n = -3$$

Q52. Show that $\bar{F} = (y^2 - z^2 + 3yz - 2x)\,\bar{i} + (3xz + 2xy)\bar{j} + (3xy - 2xz + 2z)\bar{k}$ is both solenoidal and irrotational.

Answer :

Given function is,

$$\overline{F} = (y^2 - z^2 + 3yz - 2x)\,\bar{i} + (3xz + 2xy)\,\bar{j} + (3xy - 2xz + 2z)\,\bar{k}$$

(i) $\bar{F}$ is solenoidal if and only if $\nabla.\overline{F} = 0$

$$\nabla.\overline{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k\right)$$

$$= \frac{\partial}{\partial x}\left(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)\right)$$

$$= 0 - 0 + 0 - 2 + 0 + 2x + 0 - 2x + 2 = 0$$

$$\therefore \nabla.\overline{F} = 0 \quad \text{... (1)}$$

(ii) $\overline{F}$ is irrotational if $\nabla \times \overline{F} = 0$

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right) - j\left(\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right)$$

$$+ k\left(\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right)$$

$$= i((3x - 0 + 0) - (3x + 0)) - j((3y - 2z + 0) - (0 - 2z + 3y - 0)) + k((3z + 2y) - (2y - 0 + 3z - 0))$$

$$= i\,(3x - 3x) - j(3y - 2z + 2z - 3y) + k(3z + 2y - 2y - 3y)$$

$$= i(0) - j(0) + k(0)$$

$$= 0$$

$$\therefore \nabla \times \overline{F} = 0 \quad \text{... (2)}$$

From equations (1) and (2)

$\overline{F}$ is both solenoidal and irrotational.

Q53. Show that the vector function $\vec{V} = (x^2 - yz)\,\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ is irrotational and find its scalar potential.

Answer : Dec.-16, Q15(a)

Given vector function is,

$$\overline{V} = \left(x^2 - yz\right)i + \left(y^2 - 3x\right)j + \left(z^2 - xy\right)k \quad \text{... (1)}$$

$$\text{Curl } V = \nabla \times \overline{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right] - j\left[\frac{\partial}{\partial x}(z^2 - xy) - \frac{\partial}{\partial z}(x^2 - yz)\right] + k\left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]$$

$$= i[(-x) - (-x)] - j[(-4) - (-4)] + k[-z - (-z)]$$

$$= i[-x + x] - j[-y + y] + k[-z + z] = 0i + 0j + 0k = 0.$$

$\therefore$ Curl $V = 0$

$\therefore$ V is irrotational.

But, $V = \nabla\phi$

$$\Rightarrow \quad V = \frac{i\partial\phi}{\partial x} + \frac{j\partial\phi}{\partial y} + \frac{k\partial\phi}{\partial z} \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$\frac{\partial\phi}{\partial x} = x^2 - yz \quad \text{... (3)}$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \quad \text{... (4)}$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \quad \text{... (5)}$$

Integrating equation (3),

$$\int \frac{\partial\phi}{\partial x} = \int (x^2 - yz)dx$$

$$\Rightarrow \quad \phi = \int x^2 dx - \int yzdx$$

$$\Rightarrow \quad \phi = \frac{x^3}{3} - xyz + c_1 \quad \text{... (6)}$$

Integrating equation (4),

$$\int \frac{\partial\phi}{\partial y} = \int (y^2 - zx)dy$$

$$\Rightarrow \quad \phi = \int y^2 dy - \int zxdy$$

$$\Rightarrow \quad \phi = \frac{y^3}{3} - xyz + c_2 \quad \text{... (7)}$$

Integrating equation (5),

$$\frac{\partial\phi}{\partial z} = \int (z^2 - xy)dz$$

$$\Rightarrow \quad \phi = \int z^2 dz - \int xydz$$

$$\Rightarrow \quad \phi = \frac{z^3}{3} - xyz + c_3 \quad \text{... (8)}$$

From equations (6), (7), (8),

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$$

$\therefore$ Scalar potential $(\phi) = \dfrac{x^3 + y^3 + z^3}{3} - xyz + c$.

5.2 LINE, SURFACE AND VOLUME INTEGRALS

Q54. If F = xyi – zj + x^2k and C is the curve x = t^2, y = 2t, z = t^3 from t = 0 to t = 1. Evaluate $\int_C F.dr$.

Answer :

Given that,

$$F = xyi - zj + x^2k$$

$$x = t^2, y = 2t, z = t^3 \text{ and } t = 0 \text{ to } 1$$

$$dx = 2t\,dt, \quad dy = 2\,dt, \quad dz = 3t^2dt \quad \text{... (1)}$$

$$\therefore \quad F = xyi - zj + x^2k$$

$$= t^2(2t)i - t^3j + (t^2)^2k$$

$$= 2t^3i - t^3j + t^4k$$

And $dr = dxi + dyj + dzk$

$$\therefore \quad \int_C F.dr = \int_0^1 (2t^3i - t^3j + t^4k)\,(dxi + dyj + dzk)$$

$$= \int_0^1 (2t^3 i.idx - t^3 j.jdy + t^4 k.kdz)$$

$$[\because \; i.i = j.j = k.k = 1]$$

$$= \int_0^1 (2t^3dx - t^3dy + t^4dz)$$

$$= \int_0^1 2t^3\,2t\,dt - t^3\,2\,dt + t^4\,3t^2dt$$

$$[\therefore \quad \text{From equation (1)}]$$

$$= \int_0^1 [4t^4 - 2t^3 + 3t^6]dt$$

$$= \left.\frac{4t^5}{5}\right|_0^1 - \left.\frac{2t^4}{4}\right|_0^1 + \left.\frac{3t^7}{7}\right|_0^1$$

$$= \frac{4}{5} - \frac{2}{4} + \frac{3}{7}$$

$$= \frac{3}{10} + \frac{3}{7}$$

$$= \frac{21+30}{70} = \frac{51}{70}$$

$$\therefore \quad \int_C F.dr = \frac{51}{70}$$

Q55. If $\overline{F} = (5xy - 6x^2)\,\overline{i} + (2y - 4z)\,\overline{j}$, evaluate $\int_c \overline{F}.dr$, along the curve c in the xy - plane given by y = x^3 from the point (1, 1) to (2, 8).

Answer : Dec.-17, Q15(b)

Given that,

$$\overline{F} = (5xy - 6x^2)\,\hat{i} + (2y - 4z)\,\hat{j}$$

Curve c is, $y = x^3$ ... (1)

(x, y) varies from (1, 1) to (2, 8)

The vector $\vec{r}$ can be written as,

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\Rightarrow \vec{r} = x\,\hat{i} + y\,\hat{j} \quad [\because \text{ In } xy \text{ plane}, z = 0]$$

$$\Rightarrow dr = dx\,\hat{i} + dy\,\hat{j}$$

$$\overline{F}.\overline{dr} = [(5xy - 6x^2)\,\hat{i} + (2y - 4z)\,\hat{j}\,] \,.\, (dx\,\hat{i} + dy\,\hat{j}\,)$$

$$= (5xy - 6x^2)\,dx + (2y - 4z)\,dy$$

$$= (5xy - 6x^2)\,dx + 2y\,dy$$

$$[\because z = 0]$$

From equation (1),

$dy = 3x^2\,dx$ ... (2)

$$\therefore \int_c \overline{F}.\overline{dr} = \int_1^2 (5x^4 - 6x^2)\,dx + 2(x^3)(3x^2)\,dx$$

$$[\because \text{From equations (1) and (2)}]$$

$$= \int_1^2 (5x^4 - 6x^2)\,dx + 6x^5\,dx$$

$$= \left[\frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6}\right]_1^2$$

$$= \frac{5(2)^5}{5} - \frac{6(2)^3}{3} + \frac{6(2)^6}{6} - \frac{5}{5} + \frac{6}{3} - \frac{6}{6}$$

$$= \frac{5(32)}{5} - \frac{6^2(8)}{3} + 64 - 1 + 2 - 1$$

$$= 80$$

$$\therefore \int_c \overline{F}.\overline{dr} = 80$$

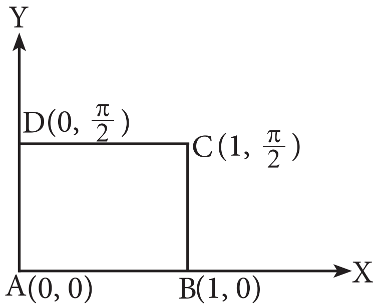
Q56. Find the circulation of $\vec{F}$ round the curve c where $\vec{F}$ = (e^x sin y)i + (e^x cos y)j and c is the rectangle whose vertices are (0, 0), (1, 0), $\left(1, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$.

Answer :

Given that,

$$\vec{F} = (e^x \sin y)\, i + (e^x \cos y) j$$

Let, $A(0, 0), B(1, 0)\ C(1, \frac{\pi}{2})$ and $D(0, \frac{\pi}{2})$ be the vertices of a rectangle as illustrated in figure.



Figure

Circulation of F is given as,

$$\int_C F.dr = \int_{AB} F.dr + \int_{BC} F.dr + \int_{CD} F.dr + \int_{DA} F.dr \quad \text{... (1)}$$

$$F.dr = \left(e^x \sin y\, i + e^x \cos y\ j\right).(dxi + dyj + dzk)$$

$$= e^x \sin y\, dx + e^x \cos y\, dy$$

Along AB:

Here, $y = 0 \Rightarrow dy = 0$

x varies from 0 to 1.

$$\int_{AB} F.dr = \int_{AB} e^x \sin(0) dx + e^1 \cos(0)(0)$$

$$= 0 + 0$$

$$= 0$$

$$\therefore \int_{AB} F.dr = 0.$$

Along BC:

Here, $x = 1 \Rightarrow dx = 0$

y varies from 0 to $\frac{\pi}{2}$.

$$\int_{BC} F.dr = \int_{BC} e^{(1)} \sin y(0) + e^1 \cos y\, dy$$

$$= e \int_0^{\frac{\pi}{2}} \cos y\, dy$$

$$= e[\sin y]_0^{\frac{\pi}{2}}$$

$$= e\left[\sin \frac{\pi}{2} - \sin 0\right]$$

$$= e[1 - 0]$$

$$= e$$

$$\therefore \int_{BC} F.dr = e.$$

Along CD

Here, $y = \frac{\pi}{2} \Rightarrow dy = 0$

x varies from 1 to 0

$$\int_{CD} F.dr = \int_{CD} \left[e^x \sin\left(\frac{\pi}{2}\right) dx + e^x \cos \frac{\pi}{2}\, 0\right]$$

$$= \int_1^0 e^x\, dx$$

$$= [e^x]_1^0$$

$$= e^0 - e^1$$

$$= 1 - e$$

$$\therefore \int_{CD} F.dr = 1 - e.$$

Along DA:

Here, $x = 0 \Rightarrow dx = 0$

y varies from $\frac{\pi}{2}$ to 0

$$\int_{DA} F.dr = \int_{DA} e^0 \sin y(0) + e^0 \cos y\, dy$$

$$= \int_{\frac{\pi}{2}}^0 \cos y\, dy$$

$$= [\sin y]_{\frac{\pi}{2}}^0$$

$$= 0 - 1$$

$$= -1$$

$$\therefore \int_{DA} F.dr = -1$$

Substituting the corresponding values in equation (1),

$$\int_C F.dr = 0 + e + 1 - e - 1 = 0$$

$$\therefore \int_C F.dr = 0.$$

Q57. Evaluate $\iint_S \overline{A}.n\, ds$ where $\overline{A} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is that part of the plane 2x + 3y + 6z = 12 which is located in the first octant.

Answer :

Given that,

$$\overline{A} = 18z\,\bar{i} - 12\bar{j} + 3y\bar{k}$$

Let, $\phi = 2x + 3y + 6z = 12$ ... (1)

$$z = \frac{12-2x-3y}{6}$$

$$\bar{A}.n = [18z\bar{i} - 12\bar{j} + 3y\bar{k}].\frac{\nabla\phi}{|\nabla\phi|} \quad ... (2)$$

Considering,

$$\nabla\phi = \left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right](2x + 3y + 6z - 12)$$

$$= 2\bar{i} + 3\bar{j} + 6\bar{k}$$

$$\therefore \quad n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{4+9+36}}$$

$$= \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7} \quad ... (3)$$

Substituting equation (3) in (2),

$$\bar{A}.n = [18z\bar{i} - 12\bar{j} + 3y\bar{k}] \times \frac{1}{7}[2\bar{i} + 3\bar{j} + 6\bar{k}]$$

$$= \frac{1}{7}[36z - 36 + 18y]$$

$$= \frac{1}{7}\left[36\frac{[12-2x-3y]}{6} - 36 + 18y\right]$$

$$\left[\because z = \frac{12-2x-3y}{6}\right]$$

$$= \frac{1}{7}[72 - 12x - 18y - 36 + 18y]$$

$$\therefore \bar{A}.n = \frac{1}{7}[36 - 12x]$$

If R be the projection of S on the xy-plane,

$$\therefore ds = \frac{dxdy}{|n.k|} = \frac{dxdy}{\left(\frac{6}{7}\right)} = \frac{7}{6}dx\,dy$$

$$\therefore \iint_S \bar{A}.n\ ds = \iint_R (\bar{A}.n)\frac{dxdy}{|n.k|}$$

$$= \iint_R \frac{1}{7}(36-12x)\frac{7}{6}dx\,dy$$

$$= \iint (6-2x)dx\,dy$$

[$\because$ From equation (1)]

$\therefore R$ of xy-plane is $2x + 3y = 12$

$3y = 12 - 2x$

$$y = \frac{12-2x}{3}$$

At $y = 0$, $x = 6$

$\therefore$ x varies from '0' to '6'

y varies from '0' to $\frac{12-2x}{3}$

$$\iint_S \bar{A}.nds = \int_{x=0}^{6}\left[\int_{y=0}^{(12-2x)/3}(6-2x)dy\right]dx$$

$$= \int_{x=0}^{6}[6[y]_{y=0}^{(12-2x)/3} - 2x[y]_{y=0}^{12-2x/3}]dx$$

$$= \int_{x=0}^{6}\left[6\frac{[12-2x]}{3} - 2x\frac{[12-2x]}{3}\right]dx$$

$$= \int_{x=0}^{6}\left[24 - 4x - 8x + \frac{4x^2}{3}\right]dx$$

$$= \int_{x=0}^{6}\left[\frac{4x^2}{3} - 12x + 24\right]dx$$

$$= 4\int_{x=0}^{6}\frac{x^2}{3}dx - 12\int_{x=0}^{6}x\,dx + 24\int_{x=0}^{6}dx$$

$$= \frac{4}{3}\left[\frac{x^3}{3}\right]_{x=0}^{6} - 12\left[\frac{x^2}{2}\right]_{x=0}^{6} + 24[x]_{x=0}^{6}$$

$$\iint_S \bar{A}.n\ ds = \frac{4}{9}[6^3] - \frac{12}{2}[36] + 24[6]$$

$$= \frac{4\times(216)}{9} - 6(36) + 24(6)$$

$$= 96 - 216 + 144 = 24$$

$$\therefore \iint_S \bar{A}.n\ ds = 24$$

Q58. Evaluate $\int\int_S \vec{F}.\hat{n}ds$ where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$ and S is the portion of the plane 2x + 3y + 6z = 12 in the first octant.

Answer : **June/July-17, Q15(b)**

Given that,

$$\bar{F} = 6zi - 4\bar{j} + y\bar{k}$$

Plane, $2x + 3y + 6z = 12$

Let,

$$\phi = 2x + 3y + 6z - 12$$

$$\nabla\phi = i\frac{\partial}{\partial x})(2x + 3y + 6z - 12) + j\frac{\partial}{\partial y}(2x + 3y + 6z - 12) + k\frac{\partial}{\partial z}(2x + 3y + 6z - 12)$$

$= i(2) + j(3) + k(6)$

$= 2i + 3j + 6k$

$\therefore$ Unit normal, $\bar{n} = \dfrac{2i + 3j + 6k}{\sqrt{2^2 + 3^2 + 6^2}}$

$= \dfrac{2i + 3j + 6k}{\sqrt{49}} = \dfrac{2i + 3j + 6k}{7}$

If R is the portion of S on xy-plane, then,

$$\therefore \quad \int_S\int \bar{F}.\bar{n}ds = \int_R\int \bar{F}.\bar{n}\frac{dx.dy}{1\bar{n}.\bar{k}1}$$

Now,

$\bar{F}.\bar{n} = (6zi - 4j + yk)\dfrac{(2i + 3j + 6k)}{7} = \dfrac{12z - 12 + 6y}{7}$

$\bar{n}.\bar{k} = \left[\dfrac{2i + 3j + 6k}{7}\right].\bar{k}$

$= \dfrac{6}{7}$

$|\bar{n}.\bar{k}| = \dfrac{6}{7}$

Limits of x and y in R-plane.

$2x + 3y + 6z = 12$

$\therefore$ R of xy-plane is,

$2x + 3y = 12$

$\Rightarrow \quad y = \dfrac{12 - 2x}{3}$

If y = 0

$12 - 2x = 0$

$\Rightarrow \quad x = 6$

Thus 'x' varies from 0 to 6 and y varies from 0 to $\left(\dfrac{12-2x}{3}\right)$

$\therefore$ Surface integral, $\displaystyle\int_S\int \bar{F}.\bar{n}ds = \int_R\int \bar{F}.\bar{n}\frac{dxdy}{1\bar{n}.\bar{k}1}$

$$= \int_0 \int_0^{\left(\frac{12-2x}{3}\right)} \frac{(12z - 12 + 6y)}{7}\frac{7}{6}dx\,dy$$

$$= \int_0 \int_0^{\left(\frac{12-2x}{3}\right)} (2z - 2 + y)\,dx\,dy$$

$$= \int_0^6 \int_0^{\left(\frac{12-2x}{3}\right)} \left[2\left(\frac{12z - 12 + 3y}{6}\right) - 2 + y\right]dx\,dy \qquad [\because \text{From equation (2)}]$$

$$= \int_0^6 \int_0^{\left(\frac{12-2x}{3}\right)} \left(4 - \frac{2}{3}x - y - 2 + y\right)dx\,dy$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} \left(2 - \frac{2}{3}x\right) dx\, dy$$

$$= \int_0^6 \left[2 - \frac{2}{3}x\right] dx \int_0^{\frac{12-2x}{3}} dy$$

$$= \int_0^6 \left(2 - \frac{2x}{3}\right)[y]_0^{\frac{12-2x}{3}}$$

$$= \frac{2}{3}\int_0^6 [3-x][6-x]\, dx$$

$$= \frac{2}{3}\int_0^6 [x^2 - 9x + 18]\, dx$$

$$= \frac{2}{3}\left[\frac{x^3}{3} - 9\frac{x^2}{2} + 18x\right]_0^6$$

$$= \frac{2}{3}\left[\frac{216}{3} - 9\frac{(36)}{2} + 18\times 6\right]$$

$$= \frac{2}{3}[72 - 162 + 108]$$

$$= \frac{2}{3}[18] = 12$$

$$\therefore \quad \int\int_S \bar{F}.\bar{n}ds = 12$$

Q59. Find the volume of the solid in the first octant bounded by the paraboloid, z = 36 – 4x² + z² – 9y².

Answer :

Given that,

$$z = 36 - 4x^2 + z^2 - 9y^2$$

Consider,

$$z = 0 \text{ to } 36 - 4x^2 - 9y^2$$

$$z = 36 - 4x^2 - 9y^2$$

$$z = 0$$

$$9y^2 = 36 - 4x^2$$

$$y = \frac{1}{3}\sqrt{36 - 4x^2}$$

$$\therefore \quad y = 0 \text{ to } \frac{1}{3}\sqrt{36 - 4x^2}$$

Substituting $y = 0$, $z = 0$

$$4x^2 = 36$$

$$x^2 = 9 \Rightarrow x = 3$$

$$\Rightarrow \quad x = 0 \text{ to } 3$$

$$V = \int_0^3 \int_0^{\frac{1}{3}\sqrt{36-4x^2}} \int_0^{36-4x^2-9y^2} dz\, dy\, dx$$

$$= \int_0^3 \left[\int_0^{\frac{1}{3}\sqrt{36-4x^2}} (36 - 4x^2 - 9y^2)\, dy\right] dx$$

$$= \int_0^3 \left[(36 - 4x^2)y - \frac{9y^3}{3}\right]_0^{\frac{1}{3}\sqrt{36-4x^2}} dx$$

$$= \int_0^3 \left[(36 - 4x^2)y - 3y^3\right]_0^{\frac{1}{3}\sqrt{36-4x^2}} dx$$

$$= \int_0^3 \left[(36 - 4x^2)\frac{1}{3}\sqrt{36 - 4x^2} - \frac{3}{27}(36 - 4x^2)^{3/2}\right] dx$$

$$= \int_0^3 \left[\frac{1}{3}(36 - 4x^2)^{3/2} - \frac{1}{9}(36 - 4x^2)^{3/2}\right] dx$$

$$= \int_0^3 \left[(36 - 4x^2)^{3/2} \times \frac{2}{9}\right] dx$$

$$= \frac{2}{9}\int_0^3 (36 - 4x^2)^{3/2}\, dx$$

$$= \frac{2}{9}\int_0^3 (4(9 - x^2))^{3/2}\, dx$$

$$= \frac{2}{9}\int_0^3 (2^2)^{3/2}(9 - x^2)^{3/2}\, dx$$

$$= \frac{2}{9}\int_0^3 8(9 - x^2)^{3/2}\, dx$$

$$= \frac{16}{9}\int_0^3 (9 - x^2)^{3/2}\, dx$$

$$V = \frac{16}{9}\int_0^3 (9 - x^2)^{3/2}\, dx$$

Let, $x = 3 \sin \theta \Rightarrow dx = 3 \cos \theta\, d\theta$

$$x = 3\sin\theta \Rightarrow 0 = 3\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$$

$$x = 3\sin\theta \Rightarrow 3 = 3\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$V = \frac{16}{9}\int_0^{\frac{\pi}{2}} (9 - 9\sin^2\theta)^{3/2} 3\cos\theta\, d\theta$$

$$= \frac{16}{9}\int_0^{\frac{\pi}{2}} (3^2)^{3/2}(1 - \sin^2\theta)^{3/2} 3\cos\theta\, d\theta$$

$$= \frac{16}{9}(3)^4 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 144\left(\frac{4-1}{4-0}\right)\left(\frac{4-3}{4-2}\right)\left(\frac{\pi}{2}\right)$$

$$= 144 \times \frac{3\pi}{16}$$

$$= 27\pi$$

$\therefore\ V = 27\pi$ cubic units

5.3 GREEN'S THEOREM IN A PLANE, GAUSS'S DIVERGENCE THEOREM, STOKES THEOREM (WITHOUT PROOFS) AND THEIR VERIFICATION

Q60. State Greens theorem in plane and apply the theorem to evaluate $\oint_C x^2 y dx + y^3 dy$, where C is the closed path formed by y = x, y = x³ from (0,0) to (1, 1).

Answer :

Green's theorem in a Plane

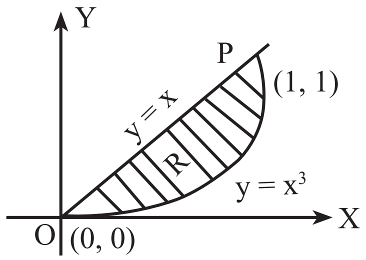
For answer refer Unit-5, Q27.

Given integral is,

$$\oint_C x^2 y dx + y^3 dy$$

Where, C is the closed path formed by the lines $y = x$, $y = x^3$

The points are $O(0, 0)$ and $P(1, 1)$ are shown in figure.



Figure

From Green's theorem,

$$\int M dx + N dy = \int\int \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\, dy \quad \text{... (1)}$$

$M = x^2 y \quad ; \quad N = y^3$

$\frac{\partial M}{\partial y} = x^2 \ ; \quad \frac{\partial N}{\partial x} = 0$

Consider,

$y = x, y = x^3$

$\Rightarrow\ x = x^3$

$\Rightarrow\ x^3 - x = 0$

$\Rightarrow\ x(x^2 - 1) = 0$

$\Rightarrow\ x = 0, x^2 = 1$

$\Rightarrow\ x = 0, x = 1$

$\therefore\ x$ varies from 0 to and y varies from x^3 to x.

Substituting the corresponding values in equation (1),

$$\int M dx + N dy = \int_{x=0}^{1} \int_{y=x^3}^{x} (0 - x^2)\, dx\, dy$$

$$= -\int_0^1 \int_{x^3}^{x} x^2\, dx\, dy$$

$$= -\int_0^1 x^2 \left[\int_{x^3}^{x} dy\right] dx$$

$$= -\int_0^1 x^2 [y]_{x^3}^{x} dx$$

$$= -\int_0^1 x^2 [x - x^3] dx$$

$$= -\int_0^1 (x^3 - x^5) dx$$

$$= -\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$$

$$= -\left[\frac{1}{4} - \frac{1}{6}\right] = -\left[\frac{1}{12}\right] = \frac{-1}{12}$$

$$\therefore \oint_C x^2 y dx + y^3\, dy = \frac{-1}{12}$$

Q61. Using Green's theorem, find the area of the region in the first quadrant bounded by the curves y = x, $y = \frac{1}{x}$, $y = \frac{x}{4}$.

Answer :

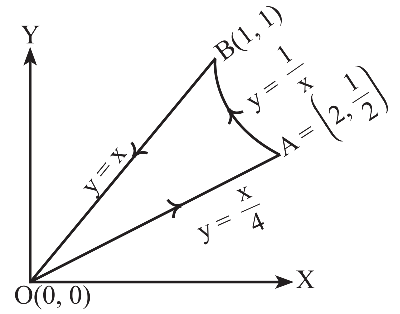
Given curves are,

$c_1 \to y = x$

$c_2 \to y = \frac{1}{x}$

$c_3 \to y = \frac{x}{4}$

The area of the region bounded by the curves in the first quadrant is shown in figure,



Figure

The area bounded by the closed curve 'c' is given as,

$$A = \frac{1}{2}\oint_c (xdy - ydx)$$

$$\Rightarrow \quad A = \frac{1}{2}\left[\oint_{c_1}(xdy - ydx) + \oint_{c_2}(xdy - ydx) + \oint_{c_3}(xdy - ydx)\right] \quad \text{... (1)}$$

Consider,

$c_1 : y = x$

$dy = dx$

$$\oint_{c_1}(xdy - ydx) = \oint_{c_1}(x.dx - x.dx)$$

$$= \oint_{c_1}(0)$$

$$= 0$$

Consider,

$c_2 : y = \frac{1}{x}$

$dy = \frac{-1}{x^2}dx$

$$\oint_{c_2}(xdy - ydx) = \oint_2^1\left(x\left(\frac{-1}{x^2}\right)dx - \frac{1}{x}dx\right)$$

$$= \oint_2^1\left(\frac{-1}{x}dx - \frac{1}{x}dx\right)$$

$$= \oint_2^1\left(\frac{-2}{x}\right)dx$$

$$= -2\oint_2^1\left(\frac{1}{x}\right)dx$$

$$= -2[\log x]_2^1$$

$$= -2\,[\log 1 - \log 2]$$

$$= -2(0 - \log 2)$$

$$= 2\log 2$$

Consider, $c_3 : y = \frac{x}{4}$

$$dy = \frac{1}{4}dx$$

$$\oint_{c_3} (xdy - ydx) = \int_1^0 \left(x.\frac{1}{4}dx - \frac{x}{4}dx\right)$$

$$= \int_1^0 \left(\frac{x}{4}dx - \frac{x}{4}dx\right)$$

$$= \int_1^0 (0) = 0$$

Substituting the corresponding values in equation (1),

$$A = \frac{1}{2}[0 + 2\log 2 + 0] = \log 2$$

$\therefore$ Area of the region = log2 square units.

Q62. Use Green's theorem to evaluate the line integral $\oint_c (xy + x^2)dx + (x^2 + y^2)dy$, where C is the closed curve of the region bounded by y = x and y = x^2.

Answer : **Dec.-16, Q15(b)**

Given integral is,

$$\int_C (xy + x^2)dx + (x^2 + y^2)dy \quad \text{... (1)}$$

By Green's theorem,

$$\int Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]dxdy \quad \text{... (2)}$$

Comparing equation (1) with L.H.S. of equation (2);

$M = xy + x^2$; $N = x^2 + y^2$

Given curves are,

$y = x$ and $y = x^2$

$\Rightarrow x = x^2$

$\Rightarrow x^2 - x = 0$

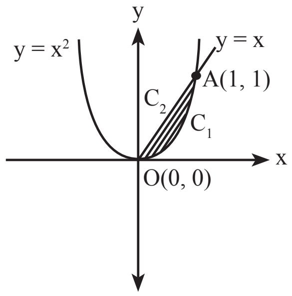
$\Rightarrow x(x - 1) = 0$

$\Rightarrow x = 0, x = 1$

If $x = 0$, then $y = 0 \Rightarrow (0, 0)$

If $x = 1$, then $y = 1 \Rightarrow (1, 1)$

$\therefore$ The intersecting points are (0, 0), (1, 1) as shown in figure below,



Figure

From figure,

$$\int Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy \quad \text{... (3)}$$

(i) Along C_1 i.e., $\overline{OA}$

$x = 0$ to 1

$y = x^2$

$dy = 2xdx$

$$\int_{C_1} Mdx + Ndy = \int_{C_1} (xy + x^2)dx + (x^2 + y^2)2xdx$$

$$= \int_0^1 (x(x^2) + x^2)dx + (x^2 + (x^2)^2)2xdx$$

$$= \int_0^1 (x^3 + x^2)dx + (x^3 + x^5)2.dx$$

$$= \int_0^1 x^3 dx + \int_0^1 x^2 dx + \int_0^1 2x^3 dx + \int_0^1 2x^5 dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1 + 2.\left[\frac{x^4}{4}\right]_0^1 + 2.\left[\frac{x^6}{6}\right]_0^1$$

$$= \left[\frac{1^4}{4} - \frac{0^4}{4}\right] + \left[\frac{1^3}{3} - \frac{0^3}{3}\right] + 2\left[\frac{1^4}{4} - \frac{0^4}{4}\right] + 2\left[\frac{1^6}{6} - \frac{0^6}{6}\right]$$

$$= \left[\frac{1}{4} - 0\right] + \left[\frac{1}{3} - 0\right] + 2\left[\frac{1}{4} - 0\right] + 2\left[\frac{1}{6} - 0\right]$$

$$= \frac{1}{4} + \frac{1}{3} + \frac{2}{4} + \frac{2}{6}$$

$$= \frac{17}{12}$$

$$\therefore \int_{C_1} Mdx + Ndy = \frac{17}{12} \quad \text{... (4)}$$

(ii) Along C_2 i.e., $\overline{AO}$

$x = 1$ to 0

$y = x$

$dy = dx$

$$\int_{C_2} Mdx + Ndy = \int_{C_2} (xy + x^2)dx + (x^2 + y^2)dy$$

$$= \int_1^0 (xx + x^2)dx + (x^2 + x^2)dx$$

$$= \int_1^0 (x^2 + x^2)dx + 2x^2 dx$$

$$= \int_1^0 2x^2 dx + 2x^2 dx = \int_1^0 4x^2 dx$$

$$= 4\left[\frac{x^3}{3}\right]_1^0 = \frac{4}{3}[0^3 - 1^3] = \frac{-4}{3}$$

$$\therefore \int_{C_2} Mdx + Ndy = \frac{-4}{3} \quad \text{... (5)}$$

Substituting equations (4) and (5) in equation (3),

$$\int Mdx + Ndy = \frac{17}{12} - \frac{4}{3}$$

$$= \frac{1}{12}$$

$$\therefore \int Mdx + Ndy = \frac{1}{12} \quad \text{... (6)}$$

Since,

$$M = xy + x^2 ; \quad N = x^2 + y^2$$

$$\frac{\partial m}{\partial y} = x \quad ; \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x$$

From figure, limits of x are 0 to 1,

limits of y are $x^2 + x$.

$$\Rightarrow \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint x.dxdy$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{x} x.dy\right]dx$$

$$= \int_{x=0}^{1} [xy]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} [x.x - x.x^2]dx$$

$$= \int_{x=0}^{1} (x^2 - x^3)dx$$

$$= \int_{x=0}^{1} x^2 dx - \int_{x=0}^{1} x^3 dx = \left[\frac{x^3}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} - \frac{0}{3} - \frac{1}{4} + \frac{0}{4} = \frac{1}{12}$$

$$\therefore \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \frac{1}{12} \quad \text{... (7)}$$

From equations (6) and (7),

$$\int Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Hence, Green's theorem is verified.

Q63. State stoke's theorem and verify the theorem for $\overline{F} = (x + y)\,\overline{i} + (y + z)\,\overline{j} - x\overline{k}$ and S is the surface of the plane 2x + y + z = 2, which is in the first octant.

Answer :

Stokes theorem

For answer refer Unit-5, Q28.

Given that,

$$\overline{F} = (x+y)i + (y+z)j - xk$$

The surface of the plane $2x + y + z = 2$ lies on XY - plane (i.e., $z = 0$) forms a triangle with the lines $x = 0$, $y = 0$ and $2x + y = 2$.

By Stoke's theorem,

$$\oint_C F.dr = \iint_s \nabla \times \overline{F}.\hat{n}ds$$

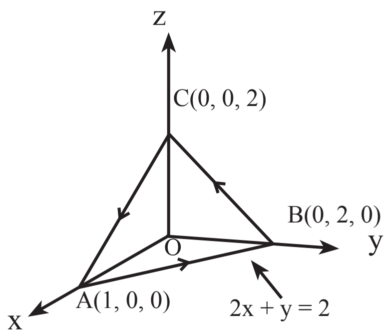


Figure (1)

Calculating the value of $\int \overline{F}.dr$ over the plane triangle as shown in figure (1)

(i) Along *AB*

x varies from 1 to 0

$z = 0$; $dz = 0$

$$2x + y + z = 2 \Rightarrow 2x + y + 0 = 2$$

$$\Rightarrow 2x + y = 2$$

$$\Rightarrow y = 2 - 2x$$

$$\Rightarrow dy = 0 - 2dx$$

$$\Rightarrow dy = -2dx$$

$$\int_{AB} \overline{F}.dr = \int_1^0 ((x+y)i+(y+z)j-xk).(dxi+dyj+dzk)$$

$$= \int_1^0 (x+y)dx+(y+z)dy-xdz$$

$$= \int_1^0 (x+2-2x)dx+(2-2x+0)(-2dx)-x(0)$$

$$= \int_1^0 (-x+2)dx+(2-2x)(-2)dx-0$$

$$= \int_1^0 (-x+2)dx+(4x-4)dx$$

$$= \int_1^0 (-x+2+4x-4)dx$$

$$= \int_1^0 (3x-2)dx$$

$$= \left[\frac{3.x^2}{2}-2x\right]_1^0$$

$$= \frac{3}{2}((0)^2-(1)^2)-2[0-1]$$

$$= \frac{3}{2}(-1)-2(-1)$$

$$= \frac{-3}{2}+2=\frac{1}{2}$$

(ii) Along *BC*

y varies from 2 to 0

$x = 0$; $dx = 0$

$2x + y + z = 2 \Rightarrow 2(0) + y + z = 2$

$\Rightarrow y + z = 2$

$\Rightarrow z = 2 - y$

$\Rightarrow dz = 0 - 2dy$

$\Rightarrow dz = -2dy$

$$\int_{BC} \overline{F}.dr = \int_2^0 ((x+y)i+(y+z)j-xk).(dxi+dyj+dzk)$$

$$= \int_2^0 (x+y)dx+(y+z)dy-xdz$$

$$= \int_2^0 (0+y)(0)+(y+2-y)dy-0.dz$$

$$= \int_2^0 0+2dy-0$$

$$= \int_2^0 2dy=(2y)_2^0$$

$$= 2(0) - 2(2) = -4$$

(iii) Along *CA*

x varies from 0 to 1

$y = 0$; $dy = 0$

$2x + y + z = 2 \Rightarrow 2x + 0 + z = 2$

$\Rightarrow 2x + z = 2$

$\Rightarrow z = 2 - 2x$

$\Rightarrow dz = 0 - 2dx$

$\Rightarrow dz = -2dx$

$$\int_{CA} \overline{F}.\overline{dr} = \int_0^1 ((x+y)i+(y+z)j-xk).(dxi+dyj+dzk)$$

$$= \int_0^1 (x+y)dx+(y+z)dy-xdz$$

$$= \int_0^1 (x+0)dx+(0+2-2x)0-x(-2dx)$$

$$= \int_0^1 xdx+0+2xdx$$

$$= \int_0^1 (x+2x)dx$$

$$= \int_0^1 3x\,dx = \left[\frac{3x^2}{2}\right]_0^1$$

$$= \frac{3}{2}((1)^2-0^2)=\frac{3}{2}$$

$$\therefore \quad \oint \overline{F}.\overline{dr} = \int_{AB} \overline{F}.dr + \int_{BC} \overline{F}.dr + \int_{CA} \overline{F}.dr$$

$$= \frac{1}{2}-4+\frac{3}{2}=-2$$

$$\therefore \quad \oint \overline{F}.\overline{dr} = -2 \qquad \text{... (1)}$$

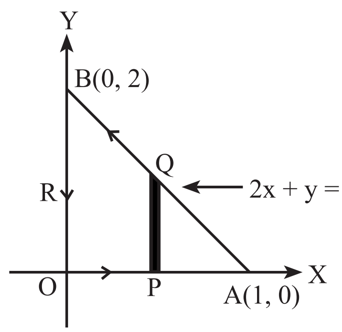


Figure (2)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(y+z)\right] - j\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial z}(x+y)\right] + k\left[\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial y}(x+y)\right]$$

$$= i[0-(0+1)] - j[-1-0] + k[0-(0+1)]$$

$$= i(-1) - j(-1) + k(-1)$$

$$= -i + j - k$$

Let, $\phi = 2x + y + z$, then

$$\hat{n}.ds = \frac{\nabla\phi}{|\nabla\phi|} \cdot \frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

$$= \frac{2i+j+k}{\sqrt{4+1+1}} \cdot \frac{dx\,dy}{\left|\frac{2i+j+k}{\sqrt{4+1+1}}.\hat{k}\right|}$$

$$= \frac{2i+j+k}{\sqrt{6}} \cdot \frac{dx\,dy}{\left|\left(\frac{2}{\sqrt{6}}i + \frac{j}{\sqrt{6}} + \frac{k}{\sqrt{6}}\right).\hat{k}\right|}$$

$$= \frac{2i+j+k}{\sqrt{6}} \cdot \frac{dx\,dy}{\left(\frac{1}{\sqrt{6}}\right)}$$

$$= (2i+j+k)\,dx\,dy$$

$$\therefore \iint_s \nabla \times \overline{F}.\hat{n}ds = \int_0^1 \int_0^{2-2x} (-i+j-k).(2i+j+k)dx\,dy$$

$$= \int_0^1 \int_0^{2-2x} (-2+1-1)dx\,dy$$

$$= \int_0^1 \int_0^{2-2x} -2\,dx\,dy$$

$$= \int_0^1 \int_0^{2-2x} (-2dy)dx$$

$$= \int_0^1 [-2y]_0^{2-2x} dx = \int_0^1 [-2(2-2x)+2(0)]dx$$

$$= \int_0^1 [-4+4x+0]dx = \int_0^1 (-4+4x)dx$$

$$= \int_0^1 \left[-4x + 4.\frac{x^2}{2}\right]_0^1 = \left[-4(1)+4(0)+\frac{4(1)^2}{2} - 4\frac{(0)^2}{2}\right]$$

$$= -4+0+2-0$$

$$= -2$$

$$\therefore \iint_s \nabla \times \overline{F}.\hat{n}ds = -2 \qquad \text{... (2)}$$

From equations (1) and (2),

$$\therefore \quad \oint_C \overline{F}.\overline{dr} = \iint_s \nabla \times \overline{F}.\hat{n}ds = -2$$

Hence, Stoke's theorem is verified.

Q64. Verify Stoke's theorem for $\overline{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes x = 0, x = a, y = 0, y = b, z = 0, z = c.

Answer :

Given that,

$$\overline{F} = (x^2 - y^2)\bar{i} + 2xy\,\bar{j} \text{ and}$$

$$x = 0, x = a, y = 0, y = b, z = 0 \text{ and } z = c$$

By Stoke's theorem,

$$\int F.dr = \int \text{curl } \overline{F}.\overline{n}ds$$

$$\text{curl } \overline{F} = \nabla.F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2) & 2xy & 0 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right] - \bar{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 - y^2)\right] + \bar{k}\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2 - y^2)\right]$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k}[2y - (-2y)]$$

$$= \bar{k}(2y + 2y) = 4\bar{k}y$$

R.H.S

$$\int \text{curl } \overline{F}.\overline{n}ds = \bar{k}(4\,y)\overline{n}.ds$$

$\because \ \bar{k}.\overline{n}ds = dx\,dy$ and R is a region on xy-plane.

$$\int \text{curl } \overline{F}.\overline{n}.ds = \int_0^a \int_0^b 4y.\,dy\,dx = \int_0^a \left[\int_0^b 4y\,dy\right] dx$$

$$= \int_0^a \left[\frac{4y^2}{2}\right]_0^b dx = \int_0^a [2y^2]_0^b\,dx$$

$$= 2\int_0^a [b^2 - 0]\,dx = 2b^2 \int_0^a dx$$

$$= 2b^2[x]_0^a = 2b^2[a - 0]$$

$$\therefore \int \text{curl } \overline{F}.n.ds = 2ab^2 = \text{R.H.S}$$

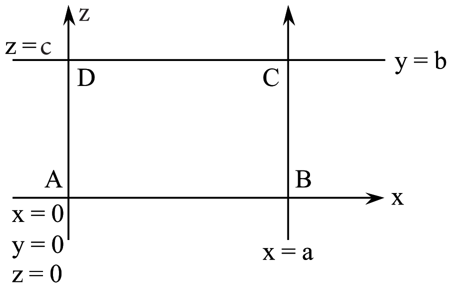
L.H.S

$$dr = \bar{i}\, dx + \bar{j}\, dy$$

$$F = (x^2 - y^2)\,\bar{i} + (2\,xy)\,\bar{j}$$

$$\int F.dr = \int [(x^2 - y^2)\bar{i} + (2xy)\bar{j}][dx\,\bar{i} + dy\,\bar{j}]$$

$$= \int [(x^2 - y^2)dx + (2xy)dy] \quad \text{... (1)}$$



Figure

Along AB axis, $y = 0$, $dy = 0$

By substituting in equation (1),

$$= \int_0^a [(x^2 - 0)dx + 0] = \int_0^a [x^2]\, dx$$

Along BC axis, $x = a$, $dx = 0$

By substituting in equation (1),

$$= \int_0^b [0 + (2\,ay)dy] = \int_0^b 2ay\, dy$$

Along CD axis, $y = b$, $dy = 0$

By substituting in equation (1),

$$= \int_0^a [(x^2 - b^2)dx + 0] = \int_0^a [(x^2 - b^2)dx]$$

Along AD axis, $x = 0$, $dx = 0$

By substituting in equation (1),

$$= \int_0^b [(0 - y^2)0 + 0] = 0$$

$$\therefore \quad \int F.dr = AB + BC - CD - AD = \int_0^a x^2.dx + \int_0^b 2ay.dy - \int_0^a (x^2 - b^2)dx - 0$$

$$= \left[\frac{x^3}{3}\right]_0^a + \left[2a\frac{y^2}{2}\right]_0^b - \left[\frac{x^3}{3}\right]_0^a + b^2[x]_0^a = \left[\frac{a^3}{3} - 0\right] + \left[2a\frac{b^2}{2} - 0\right] - \left[\frac{a^3}{3} - 0\right] + b^2[a - 0]$$

$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2$$

$\therefore$ L.H.S = R.H.S

$$\therefore \quad \int F.dr = \int \text{curl}\, \overline{F}.n.ds = 2\,ab^2$$

Hence, Stoke’s theorem is verified.

Q65. Verify Stoke's theorem for $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$ where S is the upper half surface $x^2 + y^2 + z^2 = 1$ of the sphere and C is its boundary.

Answer :

Given that,

$$F = (2x - y)\, i - y z^2 j - y^2 zk$$

Surface, $x^2 + y^2 + z^2 = 1$ in xy-plane.

By Stoke's theorem, $\int_C F.d\bar{r} = \iint_S \text{curl}\, F.\bar{x} dS$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(-y^2 z) + \frac{\partial}{\partial z}(yz^2)\right) - j\left(\frac{\partial}{\partial x}(-y^2 z) - \frac{\partial}{\partial z}(2x - y)\right) + k\left(\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial x}(2x - y)\right)$$

$$= i\,(-2yz + 2yz) - j\,(0 - 0) + k\,(0 + 1) = k$$

$$\iint_S \text{curl}\, F.\bar{n} dS = \iint_R \bar{n}.k \frac{dx\, dy}{\bar{n}.k}$$

Where, R is the projection in xy-plane,

$$\iint_R dx\, dy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dy\, dx \qquad [x^2 + y^2 = 1 \Rightarrow y = \sqrt{1 - x^2}\,]$$

$$= 4\int_{x=0}^{1} \sqrt{1 - x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= 4\left[\frac{1}{2}\sin^{-1}(1)\right]$$

$$= \frac{2\pi}{2} = \pi \qquad \ldots (1)$$

$$\int_C \bar{F}.\bar{d}r = \int_C (2x - y)\, dx \qquad (\because z = 0, \text{ i.e., } xy\text{-plane})$$

Let, $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$y = \sin\theta$

$\theta = 0$ to 2π

$$\therefore \int_C \bar{F}.\bar{d}r = \int_C (2x - y)\, dx = \int_0^{2\pi} (2\cos\theta - \sin\theta)\ (-\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} [-\sin 2\theta + \sin^2\theta] d\theta$$

$$= \frac{\cos 2\theta}{2}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{\cos 4\pi}{2} - \frac{\cos 0}{2} + \int_0^{2\pi} \frac{1}{2} d\theta - \int_0^{2\pi} \frac{\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\theta\Big|_0^{2\pi} - \frac{1}{2}\frac{\sin 2\theta}{2}\Big|_0^{2\pi}$$

$$= 0 + \frac{2\pi}{2} - \frac{1}{4}.\sin 4\pi + \frac{\sin 0}{4}$$

$$= 0 + \pi + 0 + 0 = \pi \qquad \text{... (2)}$$

$\therefore$ From equations (1) and (2)

$$\int_C F.d\bar{r} = \iint_S \text{curl}\, F.\bar{n} dS$$

Hence Stoke's theorem is verified.

Q66. Use stokes theorem to evaluate the integral $\int_c$ A.dr where, A = 2y²i + 3x²j – (2x + z)k, and C is the boundary of the triangle whose vertices are (0, 0, 0) (2, 0 0), (2, 2, 0).

Answer :

Given that,

$$\vec{A} = 2y^2\hat{i} + 3x^2\hat{j} - (2x+z)\hat{k}$$

The vertices of triangles are (0, 0, 0), (2, 0, 0), (2, 2, 0)

By stoke's theorem, $\int_c \vec{A}.\,\overrightarrow{dr} = \int\int_s (curl\, \vec{A}.\hat{n})\, ds$

$$\text{Curl} \int_c \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -(2x+z) \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(-(2x+z) - \frac{\partial}{\partial z}(3x^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-(2x+z) - \frac{\partial}{\partial z}(2y^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial y}(2y^2)\right)$$

$$= \hat{i}(0) - \hat{j}(-2) + \hat{k}(6x - 4y)$$

$$= 2\hat{j} + (6x - 4y)\hat{k}$$

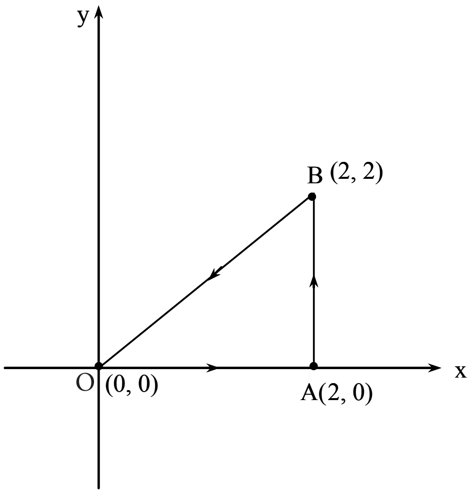
$$\therefore \text{Curl}\, \vec{A} = 2\hat{j} + (6x - 4y)\hat{k}$$

The z co-ordinate of each vertex of the triangle is zero. Therefore the triangle lies in the xy - plane as shown in the figure.

$$\therefore \hat{n} = \hat{k}$$

$$\text{Curl}\ \vec{A}.\hat{n} = (2\hat{j} + (6x - 4y)\hat{k}).\hat{k}$$

$$= 6x - 4y$$



Figure

The equation of the line OB is $y-0=\frac{2-0}{2-0}(x-0)$

$$\Rightarrow \quad y=x$$

$$\therefore \int_c \vec{A}.\vec{dr}=\iint_s (curl\,\vec{A}.\hat{n})\,ds$$

$$=\int_{x=0}^{2}\int_{y=0}^{x}(6x-4y)\,dxdy$$

$$=\int_0^2\left[\int_0^x (6x-4y)\,dy\right]dx$$

$$=\int_0^2\left(6xy-\frac{4y^2}{2}\right)_0^x dx$$

$$=\int_0^2 (6x^2-2x^2)\,dx$$

$$=\int_0^2 (4x^2)\,dx=\left(\frac{4x^3}{3}\right)_0^2$$

$$=\frac{4}{3}(8)=\frac{32}{3}$$

$$\therefore \int_c \vec{A}.\vec{dr}=\frac{32}{3}$$

Q67. State Gauss divergence theorem in plane and verify the theorem for $\bar{F}=4xz\bar{i}-y^2\bar{j}+zy\bar{k}$ over the cube x = 0, x = 1, y = 0, y = 1, z= 0, z = 1.

Answer :

For answer refer Unit-5, Q29.

Given that,

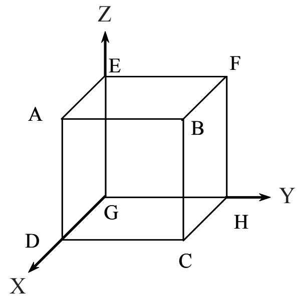
$$\overline{F}=4xzi-y^2j+zyk$$

Cube is bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

By Gauss divergence theorem,

$$\iint_s \overline{F}.nds=\iiint_v \nabla.\overline{F}dv$$

Calculating the value of $\iint_s \overline{F}.nds$ over the six faces of the cube, as shown in figure,



(i) For the face ABCD, $\hat{n}=\hat{i}$, $x=1$

$$\iint_{ABCD}\bar{F}.\hat{n}ds=\int_0^1\int_0^1 (4xzi-y^2j+zyk)\hat{i}\,dy\,dz$$

$$=\int_0^1\int_0^1 4xz.dy\,dz$$

$$=\int_0^1\int_0^1 4(1).zdz.dy \qquad [\because x=1]$$

$$=\int_0^1 4\left(\frac{z^2}{2}\right)_0^1 dy$$

$$=\int_0^1 \frac{4}{2}(1^2-0^2)\,dy$$

$$=\frac{4}{2}\int_0^1 dy$$

$$=\frac{4}{2}[y]_0^1$$

$$=\frac{4}{2}[1-0]$$

$$=2(1)=2$$

(ii) For the face HFEG, $\hat{n}=-\hat{i}$, $x=0$

$$\iint_{HFEG}\bar{F}.\hat{n}ds=\int_0^1\int_0^1 (4xzi-y^2j+zyk)(-i)\,dy\,dz$$

$$=\int_0^1\int_0^1 -4xz.dy\,dz$$

$$=-\int_0^1\int_0^1 -4(z)0.dy\,dz \quad [\because x=0]$$

$$=0$$

(iii) For the face HFBC, $\hat{n} = \hat{j}$, $y = 1$

$$\int\int_{HFBC} \bar{F}.\hat{n}ds = \int_0^1\int_0^1 (4xzi - y^2j + zyk)\hat{j}dx\,dz$$

$$= \int_0^1\int_0^1 -y^2dx\,dz$$

$$= -\int_0^1\int_0^1 (1)^2dx\,dz \qquad [\because y = 1]$$

$$= -\int_0^1\int_0^1 dxdz \qquad = -\int_0^1 [x]_0^1 dz$$

$$= -\int_0^1 (1-0)dz$$

$$= -1\int_0^1 1dz$$

$$= -1[z]_0^1$$

$$= -1(1-0) = -1$$

(iv) For the face AEGD, $\hat{n} = -\hat{j}$, $y = 0$

$$\int\int_{AEGD} \bar{F}.\hat{n}ds = \int_0^1\int_0^1 (4xzi - y^2j + zyk)(-\hat{j})dx\,dz$$

$$= -\int_0^1\int_0^1 y^2dx\,dz$$

$$= -\int_0^1\int_0^1 (0)^2dxdz \qquad [\because y = 0]$$

$$= 0$$

(v) For the face AEFB, $\hat{n} = \hat{k}$, $z = 1$

$$\int\int_{AEFB} \bar{F}.\hat{n}ds = \int_0^1\int_0^1 (4xzi - y^2j + zyk)k\,dx\,dy$$

$$= \int_0^1\int_0^1 zy.dx\,dy$$

$$= \int_0^1\int_0^1 (y\,dy)\,dx \qquad [\because z = 1]$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)_0^1 dx = \frac{1}{2}\int_0^1 (1^2 - 0^2)dx$$

$$= \frac{1}{2}\int_0^1 1dx = \frac{1}{2}[x]_0^1$$

$$= \frac{1}{2}(1-0) = \frac{1}{2}$$

(vi) For the face GHCD, $\hat{n} = -\hat{k}$, $z = 0$

$$\iint_{GHCD} \bar{F}.\hat{n}ds = \int_0^1\int_0^1 (4xzi - y^2j + zyk)(-k)\,dx\,dy$$

$$= \int_0^1\int_0^1 -zy.dx\,dy$$

$$= \int_0^1\int_0^1 (-0)y.dxdy \qquad [\because z = 0] \qquad = 0$$

$$\therefore \quad \iint \overline{F}.\hat{n}ds = \iint_{ABCD} \overline{F}.\hat{n}ds + \iint_{HFEG} \overline{F}.\hat{n}ds + \iint_{HFBC} \overline{F}.\hat{n}ds + \iint_{AEGD} \overline{F}.\hat{n}ds + \iint_{AEFB} \overline{F}.\hat{n}ds + \iint_{GHCD} \overline{F}.\hat{n}ds$$

Substituting the corresponding values in above equation,

$$\iint \overline{F}.\hat{n}ds = 2 + 0 + (-1) + 0 + \frac{1}{2} + 0$$

$$= \frac{3}{2}$$

$$\therefore \quad \iint \overline{F}.\hat{n}ds = \frac{3}{2} \qquad \text{... (1)}$$

$$\iiint_v \nabla.\bar{F}.dv = \int_0^1\int_0^1\int_0^1 \nabla.(4xzi - y^2j + zyk)\,dv$$

$$= \int_0^1\int_0^1\int_0^1 \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(4xzi - y^2j + zyk)\,\text{dv}$$

$$= \int_0^1\int_0^1\int_0^1 \left(\frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(zy)\right)dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (4z - 2y + z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (5z - 2y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left(5zy - 2.\frac{y^2}{2}\right)_0^1 dx\,dz$$

$$= \int_0^1\int_0^1 [5zy - y^2]_0^1\,dx\,dz$$

$$= \int_0^1\int_0^1 (5z(1) - (1)^2) - (5z(0) - 0^2))\,dx\,dz = \int_0^1\int_0^1 (5z - 1)\,dx\,dz$$

$$= \int_0^1 \left[5.\frac{z^2}{2} - z\right]_0^1 dx = \int_0^1 \left(\left(5\left(\frac{(1)^2}{2}\right) - (1)\right) - \left(5\left(\frac{(0)^2}{2}\right) - 0\right)\right)dx$$

$$= \int_0^1 \left(\left(5\left(\frac{1}{2}\right) - 1\right) - 0\right)dx = \int_0^1 \frac{3}{2}dx$$

$$= \frac{3}{2}\int_0^1 1dx = \frac{3}{2}[x]_0^1 = \frac{3}{2}(1 - 0) = \frac{3}{2}$$

$$\therefore \quad \iiint \nabla.\bar{F}.dv = \frac{3}{2} \qquad \text{... (2)}$$

From equations (1) and (2),

$$\int\int F.\hat{n}\, ds = \int\int_v\int \nabla.\overline{F}\ dv$$

Hence, Gauss divergence theorem is verified.

Q68. Evaluate $\int_s \overline{F}.\overline{ds}$ using Gauss divergence theorem, where $\overline{F} = 2xy\bar{i} + yz^2\bar{j} + z\bar{k}$ and S is the surface of the region bounded by x = 0, y = 0, z = 0, x + 2z = 6.

Answer :

Given that,

$$\overline{F} = 2xy\bar{i} + yz^2\bar{j} + z\bar{k}$$

S is the surfac of the region bounded by $x = 0$, $y = 0$, $x + 2z = 6$

From Gauss divergence theorem,

$$\int_S \overline{F}.ds = \int_v \nabla.\overline{F}\, dv$$

$$\Rightarrow \quad \int_S\int (2xyi + yz^2 j + zk).\bar{n}ds = \int\int_v\int \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dxdydz$$

Where, $F_1 = 2xy$, $F_2 = yz^2$, $F_3 = z$

$$\frac{\partial F_1}{\partial x} = 2y,\ \frac{\partial F_2}{\partial y} = z^2 \text{ and } \frac{\partial F_3}{\partial z} = 1$$

$$\therefore \quad \int_S\int (2xyi + yz^2 j + zk).\bar{n}ds = \int\int_v\int (2y + z^2 + 1)\, dxdydz$$

$x + 2z = 6$

When $x = 0$, $2z = 6$

$\Rightarrow \quad 2z = 6$

$\Rightarrow \quad z = 3$

$\therefore \quad z$ varies from 0 to 3

$x = 6 - 2z$

$\Rightarrow \quad x$ varies from 0 to $6 - 2z$

$x + 2z = 6$ indicates XZ plane.

If xy plane is considered, then

$x + 2y = 6$

When $x = 0$, $2y = 6$

$\Rightarrow \quad y = 3$

$\therefore \quad y$ varies from 0 to 3

$$\therefore \quad \int_S \overline{F}.\overline{d}s = \int_0^3\int_0^3\int_0^{6-2z} (2y + z^2 + 1)\, dx\, dy\, dz$$

$$= \int_0^3\int_0^3 (2y + z^2 + 1)\,[x]_0^{6-2z}\, dy\, dz$$

$$= \int_0^3\int_0^3 (2y + z^2 + 1)\,(6 - 2z - 0)\, dy\, dz$$

$$= \int_0^3\int_0^3 (2y(6 - 2z) + z^2(6 - 2z) + 6 - 2z)\, dy\, dz$$

$$= \int_0^3 \int_0^3 (12y - 4yz + 6z^2 - 2z^3 + 6 - 2z)\, dy\, dz$$

$$= \int_0^3 \left[\frac{12y^2}{2} - \frac{4y^2}{2}z + 6z^2y - 2z^3y + 6y - 2yz\right]_0^3 dz$$

$$= \int_0^3 [6(3)^2 - 2(3)^2 z + 6z^2(3) - 2z^3(3) + 6(3) - 2(3)z]\, dz$$

$$= \int_0^3 [54 - 18z + 18z^2 - 6z^3 + 18 - 6z]\, dz$$

$$= \int_0^3 (72 - 24z + 18z^2 - 6z^3)\, dz$$

$$= \left[72z - 24\frac{z^2}{2} + 18\frac{z^3}{3} - 6\frac{z^4}{4}\right]_0^3$$

$$= \left[72(3) - 24\frac{(3)^2}{2} + 18\frac{(3)^3}{3} - 6\frac{(3)^4}{4} - 0\right]$$

$$= 216 - 108 + 162 - \frac{243}{2}$$

$$= \frac{297}{2}$$

$$\therefore \quad \int_S F.ds = \frac{297}{2}$$

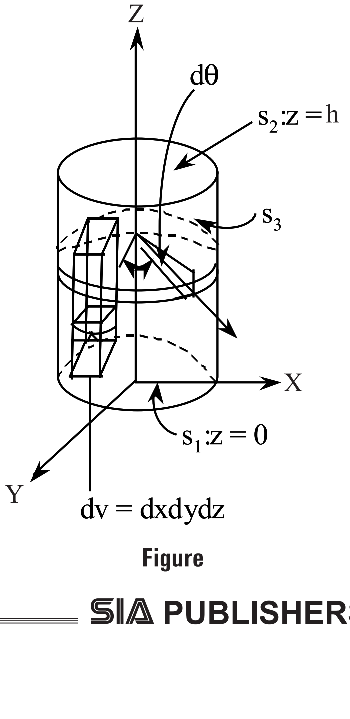
Q69. Evaluate $\iiint_V \text{div } \overline{F} dV$, where $\overline{F} = y\overline{i} + x\overline{j} + z^2\overline{k}$ over the surface of the cylinder $x^2 + y^2 = a^2$, $z = 0$ and $z = h$.

Answer :

Given vector field is,

$\overline{F} = yi + xj + z^2k$

The cylindrical region is bounded by $x^2 + y^2 = a^2$, $z = 0$ and $z = h$



Figure

$$\iiint_V \nabla.\overline{F}\,dV = \iiint_V \nabla\left(yi + xj + z^2k\right)dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(yi + xj + z^2k\right)dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h \left(\frac{\partial}{\partial x}y + \frac{\partial}{\partial y}x + \frac{\partial}{\partial z}z^2\right)dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h (0 + 0 + 2z)dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h 2z.dz.dx\,dy$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[2.\frac{z^2}{2}\right]_0^h dx\,dy$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [h^2 - 0]dx\,dy$$

$$= h^2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = h^2\int_{-a}^{a}[y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$$

$$= h^2\int_{-a}^{a}\left[\sqrt{a^2-x^2} + \sqrt{a^2-x^2}\right]dx$$

$$= h^2\int_{-a}^{a}2\sqrt{a^2-x^2}\,dx = 2h^2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$$

$$= 2h^2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{-a}^{a}$$

$$= 2h^2\left[\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right] - \left[\frac{-a}{2}\sqrt{a^2-(-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{-a}{a}\right)\right]\right]$$

$$= 2h^2\left[\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1) + \frac{a}{2}\sqrt{a^2-a^2} - \frac{a^2}{2}\sin^{-1}(-1)\right]$$

$$= 2h^2\left[0 + \frac{a^2}{2}\left(\frac{\pi}{2}\right) + \frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)\right]$$

$$= 2h^2\left[\frac{a^2\pi}{4} + 0 + \frac{a^2}{2}\left(\frac{\pi}{2}\right)\right] = 2h^2\left[\frac{a^2\pi}{4} + \frac{a^2\pi}{4}\right]$$

$$= \frac{2h^2.2a^2\pi}{4} = h^2a^2\pi$$

$$\therefore \quad \iiint_V \nabla.\overline{F}\,dV = h^2a^2\pi.$$

FACULTY OF ENGINEERING

B.E. I-Semester (Main) Examination

December - 2018

MATHEMATICS - I

Time: 3 Hours Max. Marks: 70

Note: Answer *all* questions from *Part-A* and any *five* questions from *Part-B*.

Part - A (10 × 2 = 20 Marks)

1. Determine the nature of the series $\sum_{n=1}^{\infty}\frac{2+5n}{7n-3}$ **[2] (Unit-I)**

2. Determine the nature of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ **[2] (Unit-I)**

3. Verify Rolle's mean value theorem for the function $f(x) = \frac{\sin x}{e^x}$ on $[0, \pi]$. **[2] (Unit-II)**

4. Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = a$ where α is the parameter. **[2] (Unit-II)**

5. Discuss the continuity of the function,

$$f(x, y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$ at (0, 0) **[2] (Unit-III)**

6. If $x = u(1 + v)$, $y = v(1 + u)$, then evaluate $\frac{\partial(x, y)}{\partial(u, v)}$ **[2] (Unit-III)**

7. Evaluate $\int_0^{\frac{\pi}{4}}\int_{\sin x}^{\cos x} dy\, dx$ **[2] (Unit-IV)**

8. Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} 6\, dx\, dy\, dz$ **[2] (Unit-IV)**

9. Find the unit normal vector to the surface $f(x, y, z) = x^2y - y^2z - xyz$ at $P(1, -1, 0)$. **[2] (Unit-V)**

10. If $\bar{a}$ is a constant vector and $\bar{r} = xi + yi + zk$ then evaluate $\text{div}\,(\bar{a} \times \bar{r})$ **[2] (Unit-V)**

Part - B (50 Marks)

11. (a) Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n.3^n}$. **[5] (Unit-I, Topic No. 1.2)**

(b) Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{n!}{(n+1)^n}x^n$. **[5] (Unit-I, Topic No. 1.2)**

12. (a) State and prove Cauchy's mean value theorem. **[5] (Unit-II, Topic No. 2.1)**

(b) Find the evolute of the curve $x = a\cos^3 t$, $y = a\sin^3 t$. **[5] (Unit-II, Topic No. 2.5)**

13. (a) Find the minimum value of $x + y + z$, subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 1$. **[5] (Unit-III, Topic No. 3.8)**

(b) Examine for maximum and minimum values of the function,

$f(x, y) = x^4 + 2x^{2y} - x^2 + 3y^2$. **[5] (Unit-III, Topic No. 3.8)**

14. (a) Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y)}\, dx\, dy$ by changing to polar coordinates. **[5] (Unit-IV, Topic No. 4.3)**

(b) Find the volume of the unit sphere $x^2 + y^2 + z^2 = 1$ **[5] (Unit-IV, Topic No. 4.4)**

15. Verify Green's theorem for $\oint_c (xy^2 + 2xy)\,dx + x^2 dy$ where C is the boundary of the region enclosing $y^2 = 4x$, $x = 3$.

[10] (Unit-V, Topic No. 5.3)

16. (a) Find the circle of curvature of the curve xy = 9 at the point (1, 9). **[5] (Unit-II, Topic No. 2.3)**

(b) Find the Taylor series expansion of the function $f(x, y) = \frac{1}{1 - x - y}$ around (0, 0). **[5] (Unit-II, Topic No. 2.2)**

17. (a) Evaluate $\int_0^2 \int_x^2 2y^2 \sin xy\, dy\, dx$ by changing the order of integration. **[5] (Unit-IV, Topic No. 4.2)**

(b) Using Gauss divergence theorem, evaluate $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$. Where S is the surface of the sphere $(x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 16$. **[5] (Unit-V, Topic No. 5.3)**

SOLUTIONS TO DECEMBER-2018, Q.P

Part - A (20 Marks)

Q1. Determine the nature of the series $\sum_{n=1}^{\infty} \frac{2+5n}{7n-3}$.

Answer :

Given series is,

$$\sum_{n=1}^{\infty} \frac{2+5n}{7n-3}$$

Consider, $\lim_{n\to\infty} \frac{2+5n}{7n-3}$

$$= \lim_{n\to\infty} \frac{n\left(\frac{2}{n}+5\right)}{n\left(7-\frac{3}{n}\right)}$$

$$= \lim_{\frac{1}{n}\to 0} \frac{\frac{2}{n}+5}{7-\frac{3}{n}}$$

$$= \frac{0+5}{7-0} = \frac{5}{7} < 1$$

∴ The given series is convergent.

Q2. Determine the nature of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Answer :

Given series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Expanding the above expression,

⇒ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

The above series is an alternating series with $u_n = \frac{1}{n}$

$u_n > u_{n+1} \forall n.$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = \lim_{\frac{1}{n}\to 0} \frac{1}{n} = 0$

∴ By Leibnitz's test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Q3. Verify Rolle's mean value theorem for the function f(x) = $\frac{\sin x}{e^x}$ on [0, π].

Answer :

Given function is,

$f(x) = \frac{\sin x}{e^x}$ in [0, π]

(i) To Check Continuity of f(x)

As sin$x$ and e^x are continuous functions in [0, π], $f(x)$ is also a continuous function in [0, π].

(ii) To Check Differentiability of f(x)

$f(x)$ is differentiable in $(0, \pi)$.

$\Rightarrow \quad f'(x) = \frac{\cos x - \sin x}{e^x}$

(iii) To Check f(a) = f(b)

$f(0) = \frac{\sin(0)}{e^0}$

$= 0$

$f(\pi) = \frac{\sin(\pi)}{e^\pi}$

$= 0$

$\therefore \; f(0) = f(\pi)$.

Hence, $f(x)$ satisfies all the three conditions of Rolle's theorem.

A point c exists and $c \in [0, \pi]$ such that $f'(c) = 0$.

$\Rightarrow \quad f'(c) = \frac{\cos c - \sin c}{e^c} = 0$

$\Rightarrow \quad \cos c - \sin c = 0$

$\Rightarrow \quad \cos c = \sin c$

$\Rightarrow \quad \tan c = 1$

$\Rightarrow \quad \tan c = \tan \frac{\pi}{4}$

$\Rightarrow \quad c = \frac{\pi}{4}$

$\therefore \quad c = \frac{\pi}{4}$ lies in the range $[0, \pi]$.

Hence, Rolle's theorem is verified.

Q4. Find the envelope of the family of straight lines xcos α + ysin α = a where α is the parameter.

Answer :

For answer refer Unit-II, Q27.

Q5. Discuss the continuity of the function,

$$f(x, y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \text{ at } (0, 0)$$

Answer :

Given function is,

$$f(x, y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

To check the continuity of the function $f(x, y)$ directly choose the path $y \to mx$.

$$\underset{\substack{y \to mx \\ x \to 0}}{Lt} f(x, y) = \underset{\substack{y \to mx \\ x \to 0}}{Lt} \left[\frac{(x-y)^2}{x^2+y^2}\right]$$

$$= \underset{x \to 0}{Lt} \left[\underset{y \to mx}{Lt} \frac{(x-y)^2}{x^2+y^2}\right]$$

$$= \underset{x \to 0}{Lt} \left[\frac{(x - mx)^2}{x^2 + (mx)^2} \right]$$

$$= \underset{x \to 0}{Lt} \left[\frac{x^2 + m^2x^2 - 2mx^2}{x^2 + m^2x^2} \right]$$

$$= \underset{x \to 0}{Lt} \left[\frac{x^2(1 + m^2 - 2m)}{x^2(1 + m^2)} \right]$$

$$= \underset{x \to 0}{Lt} \frac{(1 - m)^2}{(1 + m^2)}$$

i.e., the function depends on 'm' values only.

Hence, $f(x, y)$ is not continuous at (0, 0).

Q6. If x = u(1 + v), y = v(1 + u), then evaluate $\frac{\partial(x, y)}{\partial(u, v)}$.

Answer :

For answer refer Unit-III, Q18.

Q7. Evaluate $\int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\,dx$.

Answer :

Given integral is,

$$\int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\,dx$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\,dx = \int_0^{\frac{\pi}{4}} \left[\int_{\sin x}^{\cos x} dy \right] dx$$

$$= \int_0^{\frac{\pi}{4}} [y]_{\sin x}^{\cos x} dx$$

$$= \int_0^{\frac{\pi}{4}} [\cos x - \sin x]\, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$

$$= \left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right] - [\sin 0 + \cos 0]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$= \frac{2 - \sqrt{2}}{\sqrt{2}}$$

$$= \sqrt{2} - 1.$$

$$\therefore \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\,dx = \sqrt{2} - 1.$$

Q8. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 6\,dx\,dy\,dz$.

Answer :

Given integral is,

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 6\,dx\,dy\,dz$$

$$\Rightarrow \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 6\, dx\, dy\, dz = 6 \int_0^1 \int_0^{1-x} \left\{ \int_0^{1-x-y} dz \right\} dy\, dx$$

$$= 6 \int_0^1 \int_0^{1-x} [z]_0^{1-x-y}\, dy\, dx$$

$$= 6 \int_0^1 \int_0^{1-x} [1-x-y-0]\, dy\, dx$$

$$= 6 \int_0^1 \int_0^{1-x} [1-x-y]\, dy\, dx$$

$$= 6 \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= 6 \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} - 0 \right] dx$$

$$= 6 \int_0^1 \left[1 - x - x + x^2 - \left[\frac{1 + x^2 - 2x}{2} \right] \right] dx$$

$$= 6 \int_0^1 \left[1 - x - x + x^2 - \frac{1}{2} - \frac{x^2}{2} + \frac{2x}{2} \right] dx$$

$$= 6 \int_0^1 \left[\frac{1}{2} - x + \frac{x^2}{2} \right] dx$$

$$= 6 \int_0^1 \left[\frac{1 - 2x + x^2}{2} \right] dx$$

$$= \frac{6}{2} \int_0^1 (x^2 - 2x + 1)\, dx$$

$$= 3 \left[\frac{x^3}{3} - \frac{2x^2}{2} + x \right]_0^1$$

$$= 3 \left[\frac{x^3}{3} - x^2 + x \right]_0^1$$

$$= 3 \left[\frac{1}{3} - 1 + 1 - 0 \right]$$

$$= \frac{3}{3}$$

$$= 1$$

$$\therefore \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 6\, dx\, dy\, dz = 1$$

Q9. Find the unit normal vector to the surface f(x, y, z) = $x^2y - y^2z - xyz$ at P(1, –1, 0).

Answer :

Given function is,

$$f(x, y, z) = x^2y - y^2z - xyz$$

Partial derivative of 'f' with respect to x is,

$$\frac{\partial f}{\partial x} = 2xy - yz$$

Partial derivative of 'f' with respect to y is,

$$\frac{\partial f}{\partial y} = x^2 - 2yz - xz$$

Partial derivative of 'f' with respect to z is,

$$\frac{\partial f}{\partial z} = -y^2 - xy$$

Grad $f = \nabla f$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2xy - yz) + j(x^2 - 2yz - xz) + k(-y^2 - xy)$$

$$\nabla f|_{(1,-1,0)} = i(2(1)(-1) - (-1)) + j((1)^2 - 2(-1)(0) - 1(0)) + k(-(-1)^2 - 1(-1))$$

$$= i(-2 + 0) + j(1 - 0 - 0) + k(-1 + 1)$$

$$\Rightarrow \quad \nabla f|_{(1,-1,0)} = -2i + j$$

The unit normal vector is given as,

$$\frac{\nabla f}{|\nabla f|} = \frac{-2i + j}{\sqrt{(-2)^2 + (1)^2 + (0)^2}}$$

$$= \frac{-2i + j}{\sqrt{5}}$$

$\therefore$ Unit normal vector to the surface = $\dfrac{-2i + j}{\sqrt{5}}$.

Q10. If $\bar{a}$ is a constant vector and $\bar{r}$ = xi + yi + zk then evaluate div $(\bar{a} \times \bar{r})$.

Answer :

Given that,

$\bar{a}$ is a constant vector.

$$\Rightarrow \quad \bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{a} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$\therefore \quad \bar{a} \times \bar{r} = \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)$$

$$\text{div}(\bar{a} \times \bar{r}) = \frac{\partial}{\partial x}(a_2z - a_3y) - \frac{\partial}{\partial y}(a_1z - a_3x) + \frac{\partial}{\partial z}(a_1y - a_2x)$$

$$= 0 + 0 + 0 = 0$$

$$\therefore \quad \text{div}(\bar{a} \times \bar{r}) = 0$$

Part - B (50 Marks)

Q11. (a) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n.3^n}$.

Answer :

Given series is,

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n.3^n}$$

Let,

$$u_n = \frac{x^{n-1}}{n.3^n}$$

And,

$$u_{n+1} = \frac{x^{n+1-1}}{(n+1)\,3^{(n+1)}}$$

$$\Rightarrow \quad u_{n+1} = \frac{x^{n}}{(n+1)\,3^{(n+1)}}$$

Consider,

$$\frac{u_n}{u_{n+1}} = \frac{\dfrac{x^{n-1}}{n.3^n}}{\dfrac{x^n}{(n+1)\,3^{(n+1)}}}$$

$$= \frac{x^{n-1}}{n.3^n} \times \frac{(n+1)\,3^{(n+1)}}{x^n}$$

$$= \frac{(n+1)\,3^n.3}{n.3^n}\ \frac{x^n.x^{-1}}{x^n}$$

$$= \frac{(n+1)\,3}{n}.x^{-1}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{3\,(n+1)}{n}.\frac{1}{x}$$

Applying $\underset{n\to\infty}{Lt}$ on both sides,

$$\underset{n\to\infty}{Lt}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt}\ \frac{3\,(n+1)}{n}.\frac{1}{x} \qquad \text{... (3)}$$

$$= \underset{n\to\infty}{Lt}\ \frac{3n\left(1+\frac{1}{n}\right)}{n}\ \frac{1}{x}$$

$$= \underset{n\to\infty}{Lt}\ \frac{3}{x}\left(1+\frac{1}{n}\right)$$

$$= \frac{3}{x}\,(1+0) \qquad \left[\because \text{ As } n\to\infty, \frac{1}{n}\to 0\right]$$

$$\underset{n\to\infty}{Lt}\ \frac{u_n}{u_{n+1}} = \frac{3}{x}$$

By Ratio test,

Case (i)

If $\frac{3}{x} > 1$ i.e., $x < 3$, then the series is said to be convergent.

Case (ii)

If $\frac{3}{x} < 1$ i.e., $x > 3$, then the series is said to be divergent.

Case (iii)

If $\frac{3}{x} = 1$ i.e., $x = 3$, the ratio test fails.

For $x = 3$,

$$\frac{u_n}{u_{n+1}} = \frac{3(n+1)}{n} \cdot \frac{1}{3}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{n+1}{n}$$

Consider,

$$\lim_{n\to\infty}\left(n\left(\frac{u_n}{u_{n+1}} - 1\right)\right) = \lim_{n\to\infty}\left(n\left(\left(\frac{n+1}{n}\right) - 1\right)\right)$$

$$= \lim_{n\to\infty}\left(n\left(\frac{n+1-n}{n}\right)\right)$$

$$= \lim_{n\to\infty}(1) = 1$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = n.\frac{1}{n}$$ ($\because$ Multiplying by 'n' on both sides)

$$\Rightarrow \quad \underset{n\to\infty}{Lt}\left(n\left(\frac{u_n}{u_{n+1}} - 1\right)\right) = \underset{n\to\infty}{Lt}(1) = 1$$

$\because \quad \underset{n\to\infty}{lt}\left(n\left(\frac{u_n}{u_{n+1}} - 1\right)\right) = 1$, Raabe's test fails.

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n}$$

Comparing $1 + \frac{1}{n}$ with $1 + \frac{\lambda}{n}$

$\therefore$ Value of $\lambda = 1$

By Gauss test, the series diverges for $x = 3$

Hence, the given series converges for $x < 3$ and diverges for $x \geq 3$.

(b) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{(n+1)^n} x^n$.

Answer :

Given series is,

$$\sum_{n=1}^{\infty} \frac{n!}{(n+1)^n} x^n$$

Let,

$$u_n = \frac{n!}{(n+1)^n}.x^n \quad \text{... (1)}$$

$$u_{n+1} = \frac{(n+1)!}{((n+1)+1)^{n+1}}.x^{(n+1)}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{\frac{n!}{(n+1)^n}.x^n}{\left[\frac{(n+1)!}{(n+2)^{(n+1)}}\right].x^{(n+1)}}$$

$$= \frac{n!}{(n+1)^n} \times \frac{(n+2)^{(n+1)}}{(n+1)!} \times x^{n-n-1}$$

$$= \frac{n!\,(n+2)^{(n+1)}}{(n+1)!\,(n+1)^n} \cdot \frac{1}{x}$$

$$= \frac{n!\,(n+2)^{(n+1)}}{(n+1)n!\,(n+1)^n} \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{1}{x} \qquad \text{... (2)}$$

Applying $\underset{n\to\infty}{Lt}$ on both sides,

$$\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt} \frac{(n+2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{1}{x}$$

$$= \underset{n\to\infty}{Lt} \frac{(n+2)^n (n+2)}{(n+1)^n (n+1)} \cdot \frac{1}{x}$$

$$= \underset{\frac{1}{n}\to 0}{Lt} \frac{n^n\left(1+\frac{2}{n}\right)^n . n\left(1+\frac{2}{n}\right)}{xn^n\left(1+\frac{1}{n}\right)^n . n\left(1+\frac{1}{n}\right)} = \underset{\frac{1}{n}\to 0}{Lt} \frac{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{x\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$$

$$= \frac{e^2}{xe} \frac{(1+0)}{(1+0)} = \frac{e}{x}$$

By D'Alemberts ratio test,

Case (i)

If $\frac{e}{x} > 1$ i.e., $x < e$, Σu_n converges.

Case (ii)

If $\frac{e}{x} < 1$ i.e., $x > e$, Σu_n diverges

Case (iii)

If $\frac{e}{x} = 1$ i.e., $x = e$, the ratio test fails.

For $x = e$, $\frac{u_n}{u_{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{1}{e}$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n+1}\right)^{n+1} \cdot \frac{1}{e}$$

$$n \log \frac{u_n}{u_{n+1}} = n\left[\log\left(\left(\frac{n+2}{n+1}\right)^{n+1} \cdot \frac{1}{e}\right)\right]$$

$$= n\left[\log\left(\frac{n+2}{n+1}\right)^{n+1} + \log\frac{1}{e}\right]$$

$$= n\left[\log\left(\frac{n+1+1}{n+1}\right)^{n+1} - \log_e\right]$$

$$= n\left[\log\left(\frac{(n+1)\left(1+\frac{1}{(n+1)}\right)}{n+1}\right)^{n+1} - 1\right]$$

$$= n\left[\log\left(1+\frac{1}{n+1}\right)^{n+1} - 1\right]$$

$$= n\left[(n+1)\log\left(1+\frac{1}{n+1}\right) - 1\right]$$

$$= n\left[(n+1)\left[\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} \ldots\right] - 1\right]$$

$$= n\left[1 - \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} \ldots\ldots -1\right]$$

$$= n\left[\frac{-1}{2(n+1)} + \frac{1}{3(n+1)^2} \ldots\ldots\right]$$

$$= \frac{-n}{2(n+1)} + \frac{n}{3(n+1)^2} \ldots\ldots$$

$$= \frac{-n}{2n\left(1+\frac{1}{n}\right)} + \frac{n}{3n^2\left(1+\frac{1}{n}\right)^2} \ldots\ldots$$

$$= \frac{-1}{2\left(1+\frac{1}{n}\right)} + \frac{1}{3n\left(1+\frac{1}{n}\right)^2} \ldots\ldots$$

$$\underset{n\to\infty}{Lt}\ n\log\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt}\left[\frac{-1}{2\left(1+\frac{1}{n}\right)} + \frac{1}{3n\left(1+\frac{1}{n}\right)^2} \ldots\ldots\right]$$

$$= \frac{-1}{2(1+0)} + 0 \ldots\ldots$$

$$= -\frac{1}{2} < 1.$$

By Logarithm test, the series diverges for $x = e$.

Hence, the series $\sum_{n=1}^{\infty} \frac{n!}{(n+1)^n} x^n$ converges for $x < e$ and diverges for $x \geq e$.

Q12. (a) State and prove Cauchy's mean value theorem.

Answer :

Statement

If $f : [a, b] \to R$, $g : [a, b] \to R$ are such that,

(i) f and g are continuous on $[a, b]$

(ii) f, g are differentiable on (a, b) and

(iii) $g'(x) \neq 0\ \forall\ x \in (a, b)$

Then, there exists a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof

Define $\phi : [a, b] \to R$ such that

$$\phi(x) = f(x) + B\, g(x) \qquad \ldots (1)$$

Where,

$B \in R$ is given by $\phi(a) = \phi(b)$

$\Rightarrow \quad f(a) + Bg(a) = f(b) + Bg(b)$

$f(b) - f(a) = B[g(a) - g(b)]$

$$\therefore \quad B = -\left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] \quad \text{... (2)}$$

Here,

f, g are continuous on $[a, b]$,

$\phi = f + Bg$ is continuous on $[a, b]$,

f, g are differentiable on (a, b).

i.e, $\phi = f + Bg$ is also differentiable on (a, b).

$\Rightarrow \quad \phi(a) = \phi(b)$

$\therefore$ ϕ satisfies all the conditions of Rolle's theorem.

Also, there exists $c \in (a, b)$ (or) $c \in (a, b)$ such that

$\phi'(c) = 0$

$\Rightarrow f'(c) + B\, g'(c) = 0$

$$\Rightarrow B = \frac{-f'(c)}{g'(c)} \quad \text{... (3)}$$

From equations (2) and (3),

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \qquad [\because g'(c) \neq 0]$$

Hence, Cauchy's mean value theorem is proved.

(b) Find the evolute of the curve x = a cos³t, y = a sin³t.

Answer :

Given parametric equations are,

$x = a\cos^3 t,\ y = a\sin^3 t$

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\ \sin t}$$

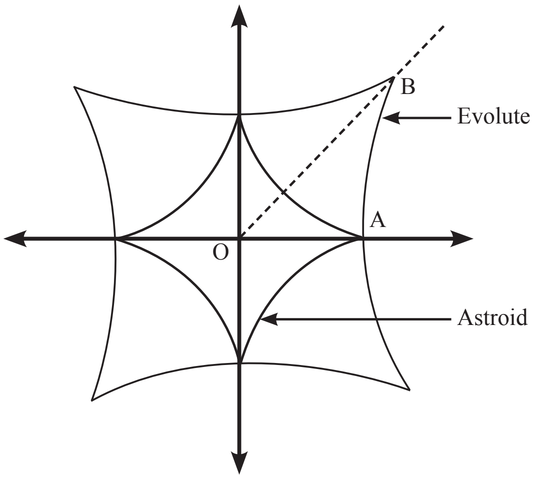
$\Rightarrow \quad y_1 = -\tan t$

$$y_2 = \frac{d}{dx}(-\tan t)$$

$$= \frac{d}{dt}(-\tan t)\frac{dt}{dx}$$

$$= -\sec^2 t \times \frac{1}{-3a\cos^2 t.\sin t}$$

$$y_2 = \frac{1}{3a\cos^4 t.\sin t}$$



Figure

The centre of curvature (x, y) is,

$$X = x - y_1 \frac{(1+y_1^2)}{y_2}$$

$$= a\cos^3 t + \tan t\,(1 + \tan^2 t).3a\cos^4 t\,\sin t$$

$$= a\cos^3 t + 3a\sin^2 t\cos t$$

Similarly,

$$Y = y + \frac{1+y_1^2}{y_2}$$

$$= a\sin^3 t + 3a\cos^2 t.\sin t$$

Consider,

$$(X+Y)^{2/3} + (X-Y)^{2/3} = (a\cos^3 t + 3a\sin^2 t\cos t + a\sin^3 t + 3a\cos^2 t\sin t)^{2/3} + (a\cos^3 t + 3a\sin^2 t\cos t - a\sin^3 t - 3a\cos^2 t\sin t)^{2/3}$$

$$= (a^{2/3}(\cos^3 t + 3\sin^2 t\cos t + \sin^3 t + 3\cos^2 t\sin t)^{2/3}) + (a^{2/3}(\cos^3 t + 3\sin^2 t\cos t - \sin^3 t - 3\cos^2 t\sin t)^{2/3})$$

$$= a^{2/3}((\cos t + \sin t)^3)^{2/3} + a^{2/3}((\cos t - \sin t)^3)^{2/3})$$

$$= a^{2/3}(\cos t + \sin t)^2 + a^{2/3}(\cos t - \sin t)^2$$

$$= a^{2/3}(\cos^2 t + \sin^2 t + 2\cos t\sin t) + a^{2/3}(\cos^2 t + \sin^2 t - 2\sin t\cos t)$$

$$= a^{2/3}(1 + 2\cos t\sin t + 1 - 2\sin t\cos t)$$

$$= a^{2/3}(1+1)$$

$$= 2a^{2/3}$$

The locus of (x, y) i.e., the evolute is, $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$.

Q13. (a) Find the minimum value of x + y + z, subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 1$.

Answer :

Given function is,

$$f(x, y, z) = x + y + z$$

$$\phi(x, y, z) = \frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 1$$

$$\Rightarrow \quad \phi(x, y, z) = \frac{a}{x}+\frac{b}{y}+\frac{c}{z} - 1$$

The auxiliary function is,

$$F(x, y, z) = x + y + z + \lambda\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right] \quad \text{... (1)}$$

Differentiating equation (1) partially with respect to x, y, z and then equating the results to zero.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x+y+z) + \lambda\,\frac{\partial}{\partial x}\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right]$$

$$= 1 + \lambda\left(\frac{-a}{x^2}\right)$$

$$= 1 - \frac{a\lambda}{x^2}$$

$$= \frac{x^2 - a\lambda}{x^2}$$

$$\therefore \quad \frac{\partial F}{\partial x} = \frac{x^2 - a\lambda}{x^2} \quad \text{... (2)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x+y+z) + \lambda\,\frac{\partial}{\partial y}\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right]$$

$$= [1] + \lambda\left(\frac{-b}{y^2}\right)$$

$$= 1 - \frac{\lambda b}{y^2}$$

$$= \frac{y^2 - b\lambda}{y^2}$$

$$\therefore \quad \frac{\partial F}{\partial y} = \frac{y^2 - b\lambda}{y^2} \quad \text{... (3)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial z}(x+y+z) + \lambda\,\frac{\partial}{\partial z}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right)$$

$$= 1 + \lambda\left(\frac{-c}{z^2}\right)$$

$$= 1 - \frac{c\lambda}{z^2}$$

$$= \frac{z^2 - c\lambda}{z^2}$$

$$\therefore \quad \frac{\partial F}{\partial z} = \frac{z^2 - c\lambda}{z^2} \quad \text{... (4)}$$

Equating equations (2), (3) and (4) to zero,

$$\frac{x^2 - a\lambda}{x^2} = 0,\ \frac{y^2 - b\lambda}{y^2} = 0,\ \frac{z^2 - c\lambda}{z^2} = 0$$

$\Rightarrow\quad x^2 - a\lambda = 0, \quad \Rightarrow y^2 - b\lambda = 0, \quad \Rightarrow z^2 - c\lambda = 0$

$\Rightarrow\quad x^2 = a\lambda, \quad \Rightarrow y^2 = b\lambda, \quad \Rightarrow z^2 = c\lambda$

$\Rightarrow\quad x = \pm\sqrt{a\lambda}, \quad \Rightarrow y = \pm\sqrt{b\lambda} \quad \Rightarrow z = \pm\sqrt{c\lambda}$

Substituting the values of x, y, z in the given constraint,

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$$

$$\Rightarrow\quad \frac{a}{\sqrt{a\lambda}} + \frac{b}{\sqrt{b\lambda}} + \frac{c}{\sqrt{c\lambda}} = 1$$

$$\Rightarrow\quad \frac{\sqrt{a}}{\sqrt{\lambda}} + \frac{\sqrt{b}}{\sqrt{\lambda}} + \frac{\sqrt{c}}{\sqrt{\lambda}} = 1$$

$$\Rightarrow\quad \frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})}{\sqrt{\lambda}} = 1$$

$$\Rightarrow\quad \sqrt{\lambda} = (\sqrt{a} + \sqrt{b} + \sqrt{c})$$

Squaring on both sides,

$$\lambda = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$$

Substituting the value of λ in x, y, z

$$x = \pm\sqrt{a}(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$$y = \pm\sqrt{b}(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$$z = \pm\sqrt{c}(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$\therefore$ The minimum value of $x + y + z = -\sqrt{a}(\sqrt{a} + \sqrt{b} + \sqrt{c}) - \sqrt{b}(\sqrt{a} + \sqrt{b} + \sqrt{c}) - \sqrt{c}(\sqrt{a} + \sqrt{b} + \sqrt{c})$

$$= -(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$$= -(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$$

$\therefore$ Minimum value $= -(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$.

(b) Examine for maximum and minimum values of the function,

$f(x, y) = x^4 + 2x^{2y} - x^2 + 3y^2$.

Answer :

Note: In the question $f(x, y) = x^4 + 2x^2y - x^2 + 3y^2$ is misprinted as $f(x, y) = x^4 + 2x^{2y} - x^2 + 3y^2$.

Given function is,

$$f(x, y) = x^4 + 2x^2y - x^2 + 3y^2 \qquad \text{... (1)}$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial f}{\partial x} = 4x^3 + 4xy - 2x \qquad \text{... (2)}$$

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial f}{\partial y} = 2x^2 + 6y \qquad \text{... (3)}$$

Differentiating equation (2) partially with respect to 'x',

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 4y - 2 = r$$

Differentiating equation (2) partially with respect to 'y',

$$\frac{\partial f}{\partial x \partial y} = 4x = s$$

Differentiating equation (3) partially with respect to 'y',

$$\frac{\partial^2 f}{\partial y^2} = 6 = t$$

The conditions for maximum or minimum value are,

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0.$$

From equation (2),

$$\frac{\partial f}{\partial x} = 0$$

$\Rightarrow \quad 4x^3 + 4xy - 2x = 0$

$\Rightarrow \quad 2x(2x^2 + 2y - 1) = 0$

$\Rightarrow \quad x = 0, 2x^2 + 2y - 1 = 0 \qquad \text{... (4)}$

From equation (3),

$$\frac{\partial f}{\partial y} = 0$$

$\Rightarrow \quad 2x^2 + 6y = 0 \qquad \text{... (5)}$

Solving equations (4) and (5),

$\therefore \quad x = \pm \frac{\sqrt{3}}{2}, y = \frac{-1}{4}$

$\therefore \quad$ The stationary points are, $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right), \left(\frac{-\sqrt{3}}{2}, \frac{-1}{4}\right)$.

Consider,

$$rt - s^2 = (12x^2 + 4y - 2)(6) - (4x)^2$$

$$= 72x^2 + 24y - 12 - 16x^2$$

$$= 56x^2 + 24y - 12$$

$$rt - s^2 \Big|_{\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)} = 56\left(\frac{\sqrt{3}}{2}\right)^2 + 24\left(\frac{-1}{4}\right) - 12$$

$$= 56\left(\frac{3}{4}\right) - \frac{24}{4} - 12$$

$$= 42 - 6 - 12$$

$$= 24 > 0.$$

$\because \quad rt - s^2 > 0, r > 0$

$$r\Big|_{\left(\frac{\sqrt{3}}{2},\frac{-1}{4}\right)} = 12\left(\frac{\sqrt{3}}{2}\right)^2 + 4\left(\frac{-1}{4}\right) - 2$$

$$= \frac{12\times3}{4} - 1 - 2$$

$$= 6 > 0$$

f has minimum value at $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$.

Minimum value is obtained by substituting $x = \frac{\sqrt{3}}{2}$, $y = \frac{-1}{4}$ in equation (1),

$$f(x, y) = f\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$$

$$= \left(\frac{\sqrt{3}}{2}\right)^4 + 2\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{-1}{4}\right) - \left(\frac{\sqrt{3}}{2}\right)^2 + 3\left(\frac{-1}{4}\right)^2$$

$$= \frac{9}{16} - \frac{3}{8} - \frac{3}{4} + \frac{3}{16}$$

$$= -\frac{3}{8}$$

$$|rt - s^2|\Big|_{\left(\frac{-\sqrt{3}}{2},\frac{-1}{4}\right)} = 56\left(\frac{-\sqrt{3}}{2}\right)^2 + 24\left(\frac{-1}{4}\right) - 12.$$

$$= 24 > 0$$

$$r\Big|_{\left(\frac{-\sqrt{3}}{2},\frac{-1}{4}\right)} = 12\left(\frac{-\sqrt{3}}{2}\right)^2 + 4\left(\frac{-1}{4}\right) - 2$$

$$= 6 > 0$$

$\therefore$ f does not have any maximum value.

Q14. (a) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y)}\, dx\, dy$ by changing to polar coordinates.

Answer :

Note: In the question $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ is misprinted as $\int_0^\infty \int_0^\infty e^{-(x^2+y)}\, dx\, dy$.

For answer refer Unit-IV, Q46, till equation (3).

(b) Find the volume of the unit sphere $x^2 + y^2 + z^2 = 1$.

Answer :

Given that,

$$x^2 + y^2 + z^2 = 1 \quad \text{... (1)}$$

$$\Rightarrow \quad z^2 = 1 - x^2 - y^2 \quad \text{... (2)}$$

$$\Rightarrow \quad z = \pm\sqrt{1 - x^2 - y^2}$$

$$\Rightarrow \quad z = +\sqrt{1 - x^2 - y^2} \text{ (or) } z = -\sqrt{1 - x^2 - y^2}$$

i.e., $-\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}$.

Let, the projection of the sphere be xy plane ($z = 0$)

From equation (1), $x^2 + y^2 = 1$

$$y^2 = 1 - x^2$$

$$\Rightarrow \quad y = \pm\sqrt{1 - x^2}$$

i.e., $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$

And $-1 \leq x \leq 1$

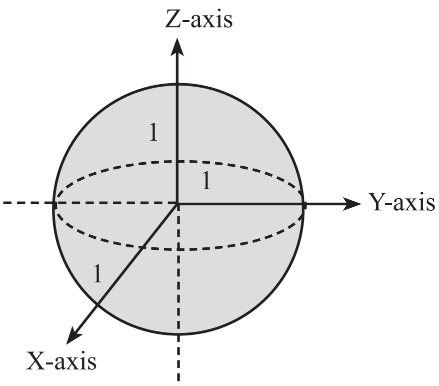


Figure (i)

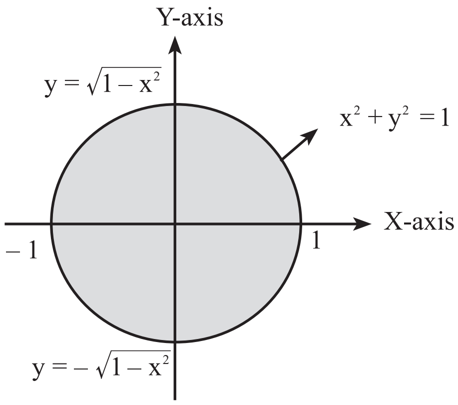


Figure (ii)

Volume of the unit sphere is given as,

$$V = \iiint_V dV$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (z)_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\, dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\sqrt{1-x^2-y^2} - \left(-\sqrt{1-x^2-y^2}\right)\right) dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy\, dx$$

$$= \int_{x=-1}^{1} \left(2 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy\right) dx$$

$$= 2 \int_{-1}^{1} \left[\int_{-\sqrt{1^2-x^2}}^{\sqrt{1^2-x^2}} \sqrt{\left(\sqrt{1^2-x^2}\right)^2 - y^2}\, dy \right] dx$$

$$= 2\int_{-1}^{1}\left[\frac{y}{2}\sqrt{1^2-y^2-x^2}+\frac{(\sqrt{1^2-x^2})^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{1^2-x^2}}\right)\right]_{-\sqrt{1^2-x^2}}^{\sqrt{1^2-x^2}} dx \quad \left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 2\int_{-1}^{1}\left[\frac{y}{2}\sqrt{1-x^2-y^2}+\frac{1-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_{-\sqrt{1^2-x^2}}^{\sqrt{1^2-x^2}} dx$$

$$= 2\int_{-1}^{1}\left[\left[\frac{\sqrt{1-x^2}}{2}\sqrt{1-x^2-(\sqrt{1-x^2})^2}+\frac{1-x^2}{2}\sin^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)\right]\right.$$

$$\left.-\left[-\frac{\sqrt{1-x^2}}{2}\sqrt{1-x^2-(-\sqrt{1-x^2})^2}+\frac{1-x^2}{2}\sin^{-1}\left(\frac{-\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)\right]\right] dx$$

$$= 2\int_{-1}^{1}\left[\left[\frac{\sqrt{1-x^2}}{2}\sqrt{1-x^2-(1-x^2)}+\frac{1-x^2}{2}\sin^{-1}(1)\right]-\left[\frac{-\sqrt{1-x^2}}{2}\sqrt{1-x^2-(1-x^2)}+\frac{1-x^2}{2}\sin^{-1}(-1)\right]\right] dx$$

$$= 2\int_{-1}^{1}\left[\left[\frac{\sqrt{1-x^2}}{2}\sqrt{1-x^2-1+x^2}+\frac{1-x^2}{2}\left(\frac{\pi}{2}\right)\right]-\left[-\frac{\sqrt{1-x^2}}{2}\sqrt{1-x^2-1+x^2}+\frac{1-x^2}{2}\left(\frac{-\pi}{2}\right)\right]\right] dx$$

$$= 2\int_{-1}^{1}\left[\left[\frac{\sqrt{1-x^2}}{2}(0)+\pi\frac{(1-x^2)}{4}\right]-\left[-\frac{\sqrt{1-x^2}}{2}(0)-\pi\left(\frac{1-x^2}{4}\right)\right]\right] dx$$

$$= 2\int_{-1}^{1}\left[0+\pi\frac{(1-x^2)}{4}+0+\pi\frac{(1-x^2)}{4}\right] dx$$

$$= 2\int_{-1}^{1}\frac{2\pi(1-x^2)}{4}\,dx$$

$$= \frac{4\pi}{4}\int_{-1}^{1}(1-x^2)dx$$

$$= \pi\int_{x=-1}^{1}(1-x^2)\,dx$$

$$= \pi\left[\int_{x=-1}^{1}dx-\int_{x=-1}^{1}x^2\,dx\right]$$

$$= \pi\left[(x)_{-1}^{1}-\left(\frac{x^3}{3}\right)_{-1}^{1}\right]$$

$$= \pi\,[(1+1)-\frac{1}{3}(1^3-(-1)^3)]$$

$$= \pi\left[2-\frac{1}{3}(2)\right]$$

$$= \pi\left[\frac{4}{3}\right] = \frac{4\pi}{3}$$

$$= \frac{4\pi}{3}$$

∴ Volume of the unit sphere is, $V = \frac{4}{3}\pi$ cc.

Q15. Verify Green's theorem for $\oint_c (xy^2 + 2xy)\,dx + x^2 dy$ where C is the boundary of the region enclosing $y^2 = 4x$, $x = 3$.

Answer :

Given integral is,

$$\int (xy^2 + 2xy)\,dx + x^2\,dy \quad \text{... (1)}$$

$$y^2 = 4x,\ x = 3$$

By Green's theorem,

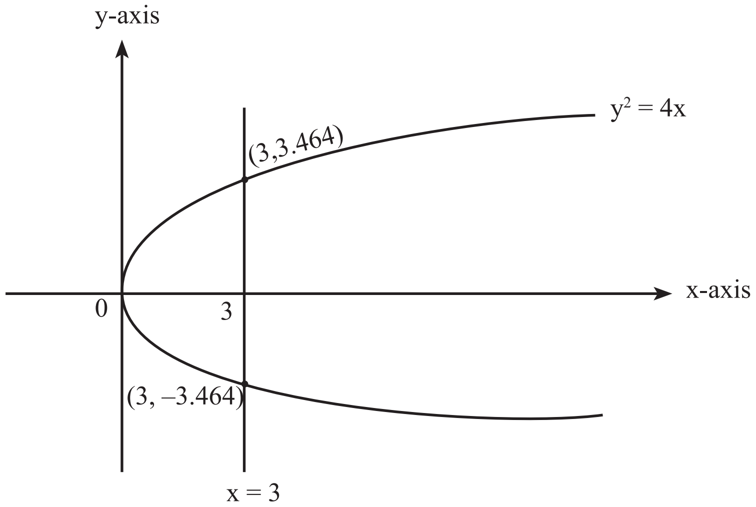
$$\int M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy \quad \text{... (2)}$$

Here, $M = xy^2 + 2xy$, $N = x^2$.

Then, $y^2 = 4(3)$

$\Rightarrow \quad y^2 = 12$

$\Rightarrow \quad y = \pm\sqrt{12}$



Figure

From figure,

$$\int M dx + \int N dy = \int_{c_1} M dx + N dy + \int_{c_2} M dx + N dy \quad \text{... (3)}$$

(i) Along c_1 i.e., $y^2 = 4x$

$$2y.dy = 4dx$$

$$\Rightarrow \quad dx = \frac{2y}{4}.dy$$

$$\Rightarrow \quad dx = \frac{y}{2}dy$$

$$\int_{c_1} M dx + N dy = \int_{c_1} (xy^2 + 2xy)\,dx + x^2\,dy$$

$$= \int_{\sqrt{12}}^{-\sqrt{12}} \left(\left(\frac{y^2}{4}.y^2 + 2.\frac{y^2}{4}.y\right)\frac{y}{2}dy + \left(\frac{y^2}{4}\right)^2 dy\right)$$

$$= \int_{\sqrt{12}}^{-\sqrt{12}} \left(\frac{y^4}{4} + \frac{y^3}{2}\right)\frac{y}{2}dy + \frac{y^4}{16}dy$$

$$= \int_{\sqrt{12}}^{-\sqrt{12}} \left(\frac{y^5}{8} + \frac{y^4}{4} + \frac{y^4}{16}\right)dy$$

$$= \left(\frac{y^6}{8\times 6} + \frac{y^5}{4\times 5} + \frac{y^5}{16\times 5}\right)_{\sqrt{12}}^{-\sqrt{12}}$$

$$= \left[\frac{y^6}{48} + \frac{y^5}{20} + \frac{y^5}{80}\right]_{\sqrt{12}}^{-\sqrt{12}}$$

$$= \frac{1}{48}((-\sqrt{12})^6 - (\sqrt{12})^6) + \frac{1}{20}((-\sqrt{12})^5 - (\sqrt{12})^5) + \frac{1}{80}((-\sqrt{12})^5 - (\sqrt{12})^5)$$

$$= \frac{1}{48}((\sqrt{12})^6 - (\sqrt{12})^6) + \frac{1}{20}(-(\sqrt{12})^5 - (\sqrt{12})^5) + \frac{1}{80}(-(\sqrt{12})^5 - (\sqrt{12})^5)$$

$$= \frac{1}{48}(0) + \frac{1}{20}(-2(\sqrt{12})^5) + \frac{1}{80}(-2(\sqrt{12})^5)$$

$$= 0 - \frac{(\sqrt{12})^5}{10} - \frac{(\sqrt{12})^5}{40}$$

$$= -18\sqrt{12}$$

$$\therefore \int_{c_1} Mdx + Ndy = -18\sqrt{12} \qquad \text{... (4)}$$

(ii) Along c_2 i.e., $x = 3$

$$dx = 0$$

$$\int_{c_2} Mdx + Ndy = \int_{c_2} (xy^2 + 2xy)dx + x^2 dy$$

$$= \int_{-\sqrt{12}}^{\sqrt{12}} (3y^2 + 6y)(0) + 9dy$$

$$= \int_{-\sqrt{12}}^{\sqrt{12}} 0 + 9dy$$

$$= 9\int_{-\sqrt{12}}^{\sqrt{12}} dy$$

$$= 9(y)_{-\sqrt{12}}^{\sqrt{12}}$$

$$= 9((\sqrt{12}) - (-\sqrt{12}))$$

$$= 9(2\sqrt{12})$$

$$= 18\sqrt{12}$$

$$\therefore \int_{c_2} Mdx + Ndy = 18\sqrt{12} \qquad \text{... (5)}$$

Substituting equations (4) and (5) in equation (3),

$$\int Mdx + Ndy = -18\sqrt{12} + 18\sqrt{12}$$
$$= 0$$

$$\therefore \quad \int Mdx + Ndy = 0 \qquad \text{... (6)}$$

Consider R.H.S of equation (1),

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy^2 + 2xy)\right) dx\, dy$$

$$= \iint_R (2x - (2xy + 2x))\, dx\, dy$$

$$= \int_{-\sqrt{12}}^{\sqrt{12}} \int_{\frac{y^2}{4}}^{3} (2x - 2xy - 2x)\, dx\, dy$$

$$= \int_{-\sqrt{12}}^{\sqrt{12}} \int_{\frac{y^2}{4}}^{3} -2xy\, dx\, dy$$

$$= -2 \int_{\frac{y^2}{4}}^{3} x \left(\int_{-\sqrt{12}}^{\sqrt{12}} y\, dy\right) dx$$

$$= -2 \int_{\frac{y^2}{4}}^{3} x \left(\frac{y^2}{2}\right)_{-\sqrt{12}}^{\sqrt{12}} dx$$

$$= -\frac{2}{2} \int_{\frac{y^2}{4}}^{3} x\left((\sqrt{12})^2 - (-\sqrt{12})^2\right) dx$$

$$= -\int_{\frac{y^2}{4}}^{3} x(12 - 12)\, dx$$

$$= -\int_{\frac{y^2}{4}}^{3} x(0)\, dx$$

$$= 0$$

$$\therefore \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = 0 \qquad \text{... (7)}$$

From equations (6) and (7),

$$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Hence, Green's theorem is verified.

Q16. (a) Find the circle of curvature of the curve xy = 9 at the point (1, 9).

Answer :

Given curve is,

$$xy = 9$$

Point, $p(x, y) = (1, 9)$

$$\Rightarrow \quad y = \frac{9}{x} \qquad \text{... (1)}$$

Differentiating equation (1) with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{9}{x}\right)$$

$$= 9\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= 9\left(\frac{-1}{x^2}\right)$$

$$\Rightarrow \quad y_1 = \frac{-9}{x^2} \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x,

$$\frac{d}{dx}(y_1) = \frac{d}{dx}\left(\frac{-9}{x^2}\right)$$

$$\Rightarrow \quad y_2 = -9\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= -9\left(\frac{-2}{x^3}\right)$$

$$\therefore \quad y_2 = \frac{18}{x^3}$$

$$\bar{x} = \frac{x - y_1(1 + y_1^2)}{y_2}$$

$$\bar{x} = x - \frac{\left(\frac{-9}{x^2}\right)\left(1 + \left(\frac{-9}{x^2}\right)^2\right)}{\frac{18}{x^3}}$$

$$\bar{x} = x + \frac{\frac{9}{x^2}\left(1 + \frac{81}{x^4}\right)}{\frac{18}{x^3}}$$

$$\bar{x} = x + \frac{9(x^4 + 81)}{x^6 \times 18} \times x^3$$

$$\bar{x} = x + \frac{(x^4 + 81)}{2x^3}$$

$$\bar{x}\Big|_{(1,9)} = 1 + \frac{(1^4 + 81)}{2(1)^3}$$

$$= 1 + \frac{(82)}{2}$$

$$= 1 + 41$$

$$= 42$$

$$\bar{y} = y + \frac{1 + y_1^2}{y_2}$$

$$= y + \frac{1 + \left(\frac{-9}{x^2}\right)^2}{\frac{18}{x^3}}$$

$$= y + \frac{1 + \frac{81}{x^4}}{\frac{18}{x^3}}$$

$$\bar{y} = y + \frac{(x^4 + 81)}{x^4.18} x^2$$

$$\bar{y} = y + \frac{(x^4 + 81)}{x^4.18} x^2$$

$$\bar{y}\,|_{(1,9)} = 9 + \frac{(1^4 + 81)}{18(1)}$$

$$= 9 + \frac{82}{18}$$

$$= \frac{122}{9}$$

Radius of curvature $(\rho) = \frac{[1 + y_1^2]^{\frac{3}{2}}}{y_2}$

$$\Rightarrow \quad \rho = \frac{\left[1 + \left(\frac{-9}{x^2}\right)^2\right]^{\frac{3}{2}}}{\frac{18}{x^3}}$$

$$= \frac{\left[1 + \frac{81}{x^4}\right]^{\frac{3}{2}}}{\frac{18}{x^3}}$$

$$= \frac{(x^4 + 81)^{\frac{3}{2}}}{(x^4)^{\frac{3}{2}}.18} .x^3$$

$$= \frac{(x^4 + 81)^{\frac{3}{2}}}{x^6.18} .x^3$$

$$= \frac{(x^4 + 81)^{\frac{3}{2}}}{18x^3}$$

$$\rho|_{(1,\,9)} = \frac{(1^4 + 81)^{\frac{3}{2}}}{18(1)^3}$$

$$= \frac{(82)^{\frac{3}{2}}}{18}$$

$$= \frac{41\sqrt{82}}{9}$$

Equation of circle of curvature is,

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

Substituting the corresponding values in the above equation,

$$\Rightarrow \quad (x - 42)^2 + \left(y - \frac{122}{9}\right)^2 = \left(\frac{41\sqrt{82}}{9}\right)^2$$

$\Rightarrow \quad (x-42)^2 + \left(y - \frac{122}{9}\right)^2 = \frac{137842}{81}$

$\therefore$ Equation of circle of curvature is,

$$(x-42)^2 + \left(y - \frac{122}{9}\right)^2 = \frac{137842}{81}$$

(b) Find the Taylor series expansion of the function f(x, y) = $\frac{1}{1-x-y}$ around (0, 0).

Answer :

Given function is,

$$f(x, y) = \frac{1}{1-x-y}$$

$$f(0, 0) = \frac{1}{1-0-0} = 1$$

$$f_x(x, y) = \frac{1}{(x+y-1)^2},\ f_x(0,0) = 1$$

$$f_{xx}(x, y) = \frac{-2}{(x+y-1)^3},\ f_{xx}(0,0) = \frac{-2}{-1} = 2$$

$$f_y(x, y) = \frac{1}{(x+y-1)^2},\ f_y(0,0) = 1$$

$$f_{yy}(x, y) = \frac{-2}{(x+y-1)^3},\ f_{yy}(0,0) = \frac{-2}{-1} = 2$$

$$f_{xy}(x, y) = \frac{\partial}{\partial x}\left(\frac{1}{(x+y-1)^2}\right)$$

$$= \frac{-2}{(x+y-1)^3}$$

$$f_{xy}(0, 0) = \frac{-2}{(-1)^3} = 2$$

$$f_{xyy}(x, y) = \frac{6}{(x+y-1)^4},\ f_{xyy}(0,0) = \frac{6}{(-1)^4} = 6$$

$$f_{xxyy}(x, y) = \frac{6}{(x+y-1)^4},\ f_{xxy}(0,0) = \frac{6}{(-1)^4} = 6$$

$$f_{xxx}(x, y) = \frac{6}{(x+y-1)^4},\ f_{xxx}(0,0) = \frac{6}{(-1)^4} = 6$$

$$f_{yyy}(x, y) = \frac{6}{(x+y-1)^4},\ f_{yyy} = 6.$$

Taylor's series at (0, 0) is given as,

$$f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\ f_{xy}(0, 0)\ y^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2 y\ f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \ldots\ldots$$

Substituting the corresponding values in the above equation,

$$f(x, y) = 1 + [x(1) + y(1)] + \frac{1}{2!}[x^2(2) + 2xy(2) + y^2(2)] + \frac{1}{3!}[x^3(6) + 3x^2y(6) + 3xy^2(6) + y^3(6)] + \ldots\ldots$$

$$= 1 + x + y + \frac{1}{2}(2x^2 + 4xy + 2y^2) + \frac{1}{6}.6\,(x^3 + 3x^2y + 3xy^2 + y^3) + \ldots\ldots$$

$$= 1 + x + y + x^2 + 2xy + y^2 + x^3 + 3xy(x + y) + y^3 + \ldots..$$

$$\therefore \quad f(x, y) = 1 + x + y + x^2 + y^2 + x^3 + y^3 + 2xy + 3xy\,(x + y).$$

Q17. (a) Evaluate $\int_0^2 \int_x^2 2y^2 \sin xy\, dy\, dx$ by changing the order of integration.

Answer :

Given integral is,

$$\int_0^2 \int_x^2 2y^2 \sin xy\, dy\, dx \quad \ldots (1)$$

The region of integration is bounded by the lines, $x = 0$ to 2 and $y = x$ to 2.

i.e., x varies from 0 to 2.

y varies from x to 2.

(a) $y = x$

If $x = 0$, $y = 0$, i.e., (0, 0)

If $x = 2$, $y = 2$ i.e., (2, 2)

Changing the order of integration, y varies from 0 to 2.

$y = x \Rightarrow x = y$ i.e.,

x varies from 0 to y.

$$\therefore \quad \int_0^2 \int_0^y 2y^2 \sin xy\, dx\, dy = \int_0^2 2y^2 \left(\int_0^y \sin xy\, dx \right) dy$$

$$= \int_0^2 2y^2 \left(\frac{-\cos xy}{y} \right)_0^y dy$$

$$= \int_0^2 -2y(\cos y(y) - \cos y(0))\, dy$$

$$= \int_0^2 -2y(\cos y^2 - 1)\, dy$$

$$= \int_0^2 (2y - 2y \cos y^2)\, dy$$

$$= 2\int_0^2 y\, dy - 2\int_0^2 y \cos y^2\, dy$$

Let, $y^2 = t \Rightarrow y = \sqrt{t}$

$\Rightarrow \quad dy = \frac{1}{2\sqrt{t}} dt$

U.L $\rightarrow$ $y = 2, t = x^2 = 4$

L.L $\rightarrow$ $y = 0, t = 0^2 = 0$

$$= 2\left(\frac{y^2}{2}\right)_0^2 - 2\int_0^4 \sqrt{t}\cos t . \frac{1}{2\sqrt{t}}\, dt$$

$$= (y^2)_{y=0}^2 - 2.\frac{1}{2}\int_{y=0}^4 \cos t\, dt$$

$$= (2^2 - 0^2) - (\sin t)_0^4$$

$$= 4 - 0 - (\sin 4 - \sin 0)$$

$$= 4 - \sin(4) + 0$$

$$= 4 - \sin(4)$$

$$= 4.75680.$$

$$\therefore \quad \int_0^2 \int_x^2 2y^2 \sin xy\, dy\, dx = 4.7568.$$

(b) Using Gauss divergence theorem, evaluate $\iint_S$ xdy dz + ydz dx + zdx dy . Where S is the surface of the sphere $(x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 16$.

Answer :

Given function is,

$$\iint_S x\, dy\, dz + y\, dz\, dx + z\, dx\, dy \qquad ... (1)$$

And sphere is,

$$(x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 16$$

$$\Rightarrow \quad (x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 4^2 \qquad ... (2)$$

Comparing equation (1) with $\iint_S (f\, dy\, dz + \phi\, dz\, dx + \psi\, dx\, dy)$

$f = x, \phi = y, \psi = z$

From Gauss divergence theorem,

$$\iint_S (f\, dy\, dz + \phi\, dz\, dx + \psi\, dx\, dy) = \iiint_V \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\, dy\, dz$$

Substituting the corresponding values in above equation,

$$\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy) = \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dx\, dy\, dz$$

$$= \iiint_V (1 + 1 + 1)\, dx\, dy\, dz$$

$$= \iiint_V 3\, dx\, dy\, dz$$

$$= 3 \iiint_V dx\, dy\, dz$$

$$= 3 \times \text{volume of sphere}$$

$$= 3 \times \frac{4}{3} \pi (4)^3$$

$$= 256\, \pi$$

$$\therefore \quad \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy) = 256\, \pi$$

Code No. 11604/A/S

FACULTY OF ENGINEERING

B.E. I-Semester (Suppl.) Examination

May/June - 2019

MATHEMATICS - I

Time: 3 Hours Max. Marks: 70

Note: Answer *all* questions from *Part-A* and any *five* questions from *Part-B*.

Part - A (10 × 2 = 20 Marks)

1. Determine the nature of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$. **(Unit-I)**

2. Determine the nature of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. **(Unit-I)**

3. Find the coefficient of x^2 in the expansion of $f(x) = e^x \sin x$. **(Unit-II)**

4. Find the radius of curvature for the curve $y^2 = x^3 + 8$ at (–2, 0). **(Unit-II)**

5. Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$ does not exist. **(Unit-III)**

6. If $xy + y^2 - 3x - 3 = 0$, then evaluate $\frac{dy}{dx}$ at (–1, 1). **(Unit-III)**

7. Evaluate $\int_0^{\pi} \int_0^{1} x \cos xy \, dy \, dx$. **(Unit-IV)**

8. Evaluate $\int_0^{\frac{\pi}{6}} \int_0^{1} \int_{-2}^{3} y \sin z \, dx \, dy \, dz$. **(Unit-IV)**

9. If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then evaluate Curl $(\vec{a} \times \vec{r})$. **(Unit-V)**

10. Evaluate $\int_C \vec{v}.dr$ where $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$ and C is the line segment from A(1, 2, 2) to B(3, 6, 6). **(Unit-V)**

Part - B (5 × 10 = 50 Marks)

11. (a) Discuss the convergence of the series $1 + \frac{1!}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + ...$ **(Unit-I, Topic No. 1.3)**

    (b) Determine the nature of the series $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$. **(Unit-I, Topic No. 1.3)**

12. (a) State and prove Lagrange's mean value theorem. **(Unit-II, Topic No. 2.1)**

    (b) Find the envelope of the family of curves $\frac{a^2}{x}\cos\theta - \frac{b^2}{y}\sin\theta$ where θ being the parameter. **(Unit-II, Topic No. 2.4)**

13. (a) Explain the method of Lagrange multipliers. Find the minimum value of the function $f(x, y) = xy + \frac{27}{x} + \frac{27}{y}$ **(Unit-III, Topic No. 3.8)**

    (b) Find the first three terms of the Taylor series of the function $f(x,y) = e^x \cos y$ around O(0,0). **(Unit-III, Topic No. 3.7)**

14. Evaluate $\int_{y=0}^{1} \int_{x=0}^{y+4} \frac{2y+1}{x+1} dx$ by changing the order of integration. **(Unit-IV, Topic No. 4.2)**

15. Verify Gauss divergence theorem for $\overline{V} = 2xy\hat{i} + 6yz\hat{j} + 3zx\hat{k}$ and D is the region bounded by the coordinate planes and the plane x + y + z = 2. **(Unit-V, Topic No. 5.3)**

16. (a) Find the extreme values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 = 1$. **(Unit-III, Topic No. 3.8)**

(b) Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(Unit-II, Topic No. 2.5)**

17. (a) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$ by changing to polar coordinates. **(Unit-IV, Topic No. 4.3)**

(b) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$, $z + 3 = x^2 + y^2$ at (–2, 1, 2). **(Unit-V, Topic No. 5.1)**

SOLUTIONS TO MAY/JUNE-2019, Q.P

Part - A (20 Marks)

Q1. Determine the nature of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

Answer :

Given series is,

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

Let $u_n = \frac{\sqrt{n}}{n^2+1}$

$$v_n = \frac{\text{Highest power of 'n' in numerator}}{\text{Highest power of 'n' in denominator}}$$

$$= \frac{n^{\frac{1}{2}}}{n^2}$$

$$= \frac{1}{n^2 . n^{\frac{-1}{2}}}$$

$$= \frac{1}{n^{\frac{3}{2}}}$$

Consider,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}.n^{\frac{3}{2}}}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}.n^{\frac{3}{2}}}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}}$$

$$= \frac{1}{1+0} = 1$$

$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0.$

$\therefore \quad \sum u_n$ and $\sum v_n$ will both converge or diverge together.

By P-test, $\sum v_n$ is convergent with $p = \frac{3}{2} > 1$.

$\because \quad \sum v_n$ is convergent, $\sum u_n$ is also convergent.

$\therefore \quad \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ is convergent.

Q2. Determine the nature of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

Answer :

For answer refer Unit-I, Page No. 1.34, Q.No. 61.

Q3. Find the coefficient of x^2 in the expansion of $f(x) = e^x \sin x$.

Answer :

Given that,

$$f(x) = e^x \sin x \qquad \text{... (1)}$$

$$f(0) = e^0 \sin 0 = 0$$

Differentiating equation (1) with respect to x

$$f'(x) = e^x(\cos x) + \sin x(e^x)$$

$$\Rightarrow \quad f'(x) = e^x\cos x + e^x\sin x \qquad \text{... (2)}$$

$$f'(0) = e^0\cos(0) + e^0\sin 0$$

$$\Rightarrow \quad f'(0) = 1(1) + 1(0) = 1$$

Differentiating equation (2) with respect to x,

$$f''(x) = e^x(-\sin x) + \cos x(e^x) + \sin x(e^x) + e^x(\cos x)$$

$$= -e^x\sin x + e^x\cos x + e^x\sin x + e^x\cos x$$

$$= 2e^x\cos x$$

$$f''(0) = 2e^0\cos(0)$$

$$= 2.1.(1)$$

$$f''(0) = 2$$

The Maclaurins series is given as,

$$f(x) = f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) \quad [\because \text{Upto } x^2 \text{ term}]$$

Substituting the corresponding values in above equation,

$$e^x\sin x = 0 + \frac{x}{1}(1) + \frac{x^2}{2!}(2)$$

$$= 0 + x + x^2$$

$$= x + x^2$$

$\therefore$ The coefficient of x^2 is 1.

Q4. Find the radius of curvature for the curve $y^2 = x^3 + 8$ at $(-2, 0)$.

Answer :

Given curve is,

$$y^2 = x^3 + 8$$

Differentiating above equation with respect to x,

$$2y.\frac{dy}{dx} = 3x^2 + 0$$

$$\Rightarrow \quad 2y.\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad y.\frac{dy}{dx} = \frac{3x^2}{2} \qquad \text{... (1)}$$

Differentiating above equation with respect to x

$$y.\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = \frac{3}{2}(2x)$$

$$\Rightarrow \quad y.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3x$$

$$\Rightarrow \quad y.\frac{d^2y}{dx^2} = 3x - \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{3x - \left(\frac{dy}{dx}\right)^2}{y}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{3x - \left(\frac{3x^2}{2y}\right)^2}{y} \quad [\because \text{ From Equation (1)}]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{3x - \frac{9x^4}{4y^2}}{y}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{12xy^2 - 9x^4}{4y^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,0)} = \frac{12(-2)(0)^2 - 9(-2)^4}{4(0)}$$

The expression for radius of curvature is given as,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \text{... (2)}$$

Substituting the corresponding values in equation (2),

$$\rho = \frac{\left[1+\left(\frac{3x^2}{2y}\right)^2\right]^{\frac{3}{2}}}{\frac{12xy^2-9x^4}{4y^3}}$$

$$= \frac{\left[1+\frac{9x^4}{4y^2}\right]^{\frac{3}{2}}}{\frac{12xy^2-9x^4}{4y^3}}$$

$$= \frac{[4y^2+9x^4]^{\frac{3}{2}}.4y^3}{(4y^2)^{\frac{3}{2}}(12xy^2-9x^4)}$$

$$= \frac{(4y^2+9x^4)^{\frac{3}{2}}.4y^3}{8y^3(12xy^2-9x^4)}$$

$$= \frac{(4y^2+9x^4)^{\frac{3}{2}}}{2(12xy^2-9x^4)}$$

$$\rho|_{(-2,0)} = \frac{(4(0)^2+9(-2)^4)^{\frac{3}{2}}}{2(12(-2)(0)^2-9(-2)^4)}$$

$$= \frac{(0+144)^{\frac{3}{2}}}{2(0-144)}$$

$$= -6$$

$\therefore$ Radius of curvature is 6.

Q5. Show that the limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$ does not exist.

Answer :

Given limit is,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

Let,

$$P_1 = \underset{\substack{x\to 0\\ y\to 0}}{Lt}\left[\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{y\to 0}{Lt}\left[\underset{x\to 0}{Lt}\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{y\to 0}{Lt}\left[\frac{(0)^3y}{0^6+y^2}\right]$$

$$= \underset{y\to 0}{Lt}\left[\frac{(0)y}{y^2}\right]$$

$$= 0$$

$\therefore \quad P_1 = 0$

Let,

$$P_2 = \underset{\substack{y\to 0\\ x\to 0}}{Lt}\left[\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\underset{y\to 0}{Lt}\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x^3(0)}{x^6+(0)^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{(0)x^3}{x^6}\right]$$

$$= 0$$

$\therefore \quad P_2 = 0$

Let,

$$P_3 = \underset{\substack{y\to mx\\ x\to 0}}{Lt}\left[\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\underset{y\to mx}{Lt}\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x^3mx}{x^6+(mx)^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{mx^4}{x^2(x^4+m^2)}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{mx^2}{(x^4+m^2)}\right]$$

$$= \frac{m(0)^2}{0+m^2}$$

$$= 0$$

$\therefore \quad P_3 = 0$

Let,

$$P_4 = \underset{\substack{y\to mx^2\\x\to 0}}{Lt}\left[\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\underset{y\to mx^2}{Lt}\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x^3(mx^2)}{x^6+(mx^2)^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x^5m}{x^4(x^2+m^2)}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{xm}{(x^2+m^2)}\right]$$

$$= \frac{(0)m}{0^2+m^2} = 0$$

$\therefore \quad P_4 = 0$

Let,

$$P_5 = \underset{\substack{y\to mx^3\\x\to 0}}{Lt}\left[\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\underset{y\to mx^3}{Lt}\frac{x^3y}{x^6+y^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x^3(mx^3)}{x^6+(mx^3)^2}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{x^6m}{x^6(1+m^2)}\right]$$

$$= \underset{x\to 0}{Lt}\left[\frac{m}{1+m^2}\right]$$

$$\therefore \quad P_5 = \underset{x\to 0}{Lt}\left[\frac{m}{1+m^2}\right]$$

Since, the value of limit along path P_5 is independent of x the limit does not exist.

Q6. If $xy + y^2 - 3x - 3 = 0$, then evaluate $\frac{dy}{dx}$ at (–1, 1).

Answer :

Given equation is,

$$xy + y^2 - 3x - 3 = 0$$

Differentiating above equation with respect to x,

$$\frac{d}{dx}(xy + y^2 - 3x - 3) = 0$$

$$\Rightarrow \left[x.\frac{dy}{dx} + y(1)\right] + 2y.\frac{dy}{dx} - 3 = 0$$

$$\Rightarrow x.\frac{dy}{dx} + y + 2y.\frac{dy}{dx} - 3 = 0$$

$$\Rightarrow x.\frac{dy}{dx} + 2y.\frac{dy}{dx} = 3 - y$$

$$\Rightarrow \frac{dy}{dx}[x + 2y] = 3 - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{3-y}{x+2y}$$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{3-1}{-1+2(1)}$$

$$= \frac{2}{1} = 2$$

$$\therefore \left.\frac{dy}{dx}\right|_{(-1,1)} = 2$$

Q7. Evaluate $\int_0^{\pi}\int_0^{1} x \cos xy\, dy\, dx$.

Answer :

Given integral is,

$$\int_0^{\pi}\int_0^{1} x \cos xy\, dy\, dx$$

$$\Rightarrow \int_0^{\pi}\int_0^{1} x\cos xy\,dy\,dx = \int_0^{\pi} x\left[\int_0^1 \cos xy\, dy\right]dx$$

$$= \int_0^{\pi} x\left[\frac{\sin xy}{x}\right]_0^1 dx$$

$$= \int_0^{\pi} \frac{x}{x}(\sin xy)_0^1 dx$$

$$= \int_0^{\pi} (\sin x(1) - \sin x(0))\,dx$$

$$= \int_0^{\pi} (\sin x - 0)\,dx$$

$$= \int_0^{\pi} \sin x\, dx$$

$$= (-\cos x)_0^{\pi}$$

$$= -(\cos\pi - \cos 0)$$

$$= -(-1-1)$$

$$= 2$$

$$\therefore \int_0^{\pi}\int_0^{1} x\cos xy\,dy\,dx = 2$$

Q8. Evaluate $\int_0^{\frac{\pi}{6}}\int_0^{1}\int_{-2}^{3} y \sin z\, dx\, dy\, dz$.

Answer :

Given integral is,

$$\int_0^{\frac{\pi}{6}}\int_0^{1}\int_{-2}^{3} y \sin z\, dx\, dy\, dz$$

$$= \int_0^{\frac{\pi}{6}} \sin z \int_0^1 y \left[\int_{-2}^{3} 1\,dx\right] dy\,dz$$

$$= \int_0^{\frac{\pi}{6}} \sin z \int_0^1 y\,[x]_{-2}^{3}\,dy\,dz$$

$$= \int_0^{\frac{\pi}{6}} \sin z \int_0^1 y[3-(-2)]dy\,dz$$

$$= \int_0^{\frac{\pi}{6}} \sin z \int_0^1 y\,(5)\,dy\,dz$$

$$= 5\int_0^{\frac{\pi}{6}} \sin z\left[\int_0^1 y\,dy\right] dz$$

$$= 5\int_0^{\frac{\pi}{6}} \sin z\left[\frac{y^2}{2}\right]_0^1 dz$$

$$= \frac{5}{2}\int_0^{\frac{\pi}{6}} \sin z[1^2-0^2]\,dz = \frac{5}{2}\int_0^{\frac{\pi}{6}} \sin z\,(1)\,dz$$

$$= \frac{5}{2}\int_0^{\frac{\pi}{6}} \sin z\,dz = \frac{5}{2}(-\cos z)_0^{\frac{\pi}{6}}$$

$$= -\frac{5}{2}\left(\cos\frac{\pi}{6} - \cos 0\right)$$

$$= -\frac{5}{2}\left(\frac{\sqrt{3}}{2} - 1\right) = -\frac{5\sqrt{3}}{4} + \frac{5}{2}$$

$$\therefore \int_0^{\frac{\pi}{6}}\int_0^1\int_{-2}^{3} y\sin z\,dx\,dy\,dz = -\frac{5\sqrt{3}}{4} + \frac{5}{2}$$

Q9. If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then evaluate Curl $(\vec{a}\times\vec{r})$.

Answer :

Given that,

$\vec{a}$ is a constant vector

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Consider,

$$\vec{a}\times\vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= i[a_2z - a_3y] - j\,[a_1z - a_3x] + k[a_1y - a_2x]$$

$$\therefore \quad \vec{a}\times\vec{r} = i[a_2z - a_3y] - j\,[a_1z - a_3x] + k[a_1y - a_2x]$$

Consider,

$$\text{Curl}\,(\vec{a}\times\vec{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}[a_1y - a_2x] - \frac{\partial}{\partial z}[a_3x - a_1z]\right]$$

$$-j\left[\frac{\partial}{\partial x}[a_1y - a_2x] - \frac{\partial}{\partial z}[a_2z - a_3y]\right]$$

$$+k\left[\frac{\partial}{\partial x}[a_3x - a_1z] - \frac{\partial}{\partial y}[a_2z - a_3y]\right]$$

$$= i[a_1 + a_1] - j[-a_2 - a_2] + k[a_3 + a_3]$$

$$= 2ia_1 + 2ja_2 + 2ja_3 = 2[a_1i + a_2j + a_3k] = 2\vec{a}$$

$$\therefore \quad \text{Curl } \vec{a}\times\vec{r} = 2\vec{a}$$

Q10. Evaluate $\int_C \vec{v}.dr$ where $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$ and C is the line segment from A(1, 2, 2) to B(3, 6, 6).

Answer :

Given that,

$V = xi + yj + zk$

Points are, $A(1, 2, 2)$ and $B(3, 6, 6)$

$r = a + t(b - a)$

$\Rightarrow \quad r = (i + 2j + 2k) + t\,(3i + 6j + 6k) - (i + 2j + 2k)$

$\Rightarrow \quad r = (i + 2j + 2k) + t\,(3i + 6j + 6k - i - 2j - 2k)$

$\Rightarrow \quad r = (i + 2i + 2k) + t(2i + 4j + 4k)$

$$\Rightarrow \quad r = \underbrace{(1+2t)i}_{x} + \underbrace{(2+4t)j}_{y} + \underbrace{(2+4t)k}_{z} \quad ...(1)$$

$$\frac{dr}{dt} = (0 + 2)i + (0 + 4)j + (0 + 4)k$$

$$\Rightarrow \quad \frac{dr}{dt} = 2i + 4j + 4k$$

$$\int_c v.dr = \int_0^1 v.\frac{dr}{dt}.dt$$

$$= \int_0^1 (xi + yj + zk)\,(2i + 4j + 4k)\,dt$$

$$= \int_0^1 (2x + 4y + 4k)\,dt$$

$$= \int_0^1 (2\,(1 + 2t) + 4\,(2 + 4t) + 4\,(2 + 4t))\,dt$$

[$\because$ From equation (1)]

$$= \int_0^1 (2 + 4t + 8 + 16t + 8 + 16t)\,dt$$

$$= \int_0^1 (18 + 36t)\,dt$$

$$= \left(18t + 36\frac{t^2}{2}\right)_0^1 = \left(18t + 18t^2\right)_0^1$$

$$= 18(1) + 18(1)^2 - 18(0) - 18(0)^2 = 36$$

$$\therefore \int_c v.dr = 36$$

Part - B (50 Marks)

Q11. (a) Discuss the convergence of the series $1+\frac{1!}{2}x+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+......$

Answer :

For answer refer Unit-I, Q52, Page No. 1.26.

(b) Determine the nature of the series $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

Answer :

Given series is,

$$\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$$

Let, $u_n=\left(\frac{n}{3n+1}\right)^n$

Applying n^{th} root on both sides,

$$\sqrt[n]{u_n}=\sqrt[n]{\left(\frac{n}{3n+1}\right)^n}$$

$$\Rightarrow \quad u_n^{\frac{1}{n}}=\left(\left(\frac{n}{3n+1}\right)^n\right)^{\frac{1}{n}}$$

$$\Rightarrow \quad u_n^{\frac{1}{n}}=\frac{n}{3n+1}$$

Applying limit on both sides

$$\lim_{n\to\infty}=\lim_{n\to\infty}\left(\frac{n}{3n-1}\right)$$

$$=\lim_{n\to\infty}\frac{n}{n\left(3+\frac{1}{n}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{3+\frac{1}{n}}$$

$$=\lim_{\frac{1}{n}\to 0}\frac{1}{3+\frac{1}{n}}$$

$$=\frac{1}{3+0}$$

$$=\frac{1}{3}<1$$

∴ By Cauchy's n^{th} root test, the given series is convergent.

$$\therefore \quad \sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n \text{ is convergent.}$$

Q12. (a) State and prove Lagrange's mean value theorem.

Answer :

Statement

If $f(x)$ is a function defined in $[a, b]$ such that,

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is derivable in (a, b)

Then, there exists atleast one point $c\in(a, b)$ such that,

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Proof

Consider a function, $\phi(x)$ defined in $[a, b]$ by,

$$\phi(x)=f(x)+Ax \qquad ...(1)$$

$$\phi(a)=\phi(b)$$

$$\Rightarrow \quad f(a)+Aa=f(b)+Ab$$

$$\Rightarrow \quad A(a)-A(b)=f(b)-f(a)$$

$$\Rightarrow \quad -A(b-a)=f(b)-f(a)$$

$$\Rightarrow \quad -A=\frac{f(b)-f(a)}{b-a} \qquad ...(2)$$

Substituting equation (2) in equation (1),

$$\phi(x)=f(x)-\left[\frac{f(b)-f(a)}{b-a}\right]x,$$

(i) Since, $f(x)$ is continuous in $[a, b]$ and x is a polynomial, $\phi(x)$ is also continuous in $[a, b]$

(ii) Since, $f(x)$ is derivable in (a, b) and x is derivable $\phi(x)$ is also derivable in (a, b)

(iii) $\phi(a)=f(a)-\left[\frac{f(b)-f(a)}{b-a}\right].\,a=af(b)-bf(a)$

$$\phi(b)=f(b)-\left[\frac{f(b)-f(a)}{b-a}\right]b=af(b)-bf(a)$$

$$\Rightarrow \quad \phi(a)=\phi(b)$$

∴ $\phi(x)$ satisfies all the conditions of Rolle's theorem.

Then, there exists at least one point $c\in(a, b)$ such that $\phi'(c)=0$

$$\therefore \quad \phi'(x)=f'(x)+A$$

$$\Rightarrow \quad \phi'(c)=0 \Rightarrow f'(c)+A=0$$

$$\Rightarrow \quad f'(c)=-A$$

$$\therefore \quad f'(c)=\frac{f(b)-f(a)}{b-a} \qquad [\because \text{From equation (2)}]$$

(b) Find the envelope of the family of curves $\frac{a^2}{x}\cos\theta-\frac{b^2}{y}\sin\theta$ where θ being the parameter.

Answer :

Given curve is,

$$\frac{a^2}{x}\cos\theta-\frac{b^2}{y}\sin\theta=c \qquad ...(1)$$

Differentiating equation (1) with respect to 'θ',

$$\frac{a^2}{x}(-\sin\theta)-\frac{b^2}{y}(\cos\theta)=0$$

$$\Rightarrow \quad \frac{-a^2}{x}\sin\theta-\frac{b^2}{y}\cos\theta=0$$

$$\Rightarrow \quad \frac{a^2}{x}\sin\theta+\frac{b^2}{y}\cos\theta=0 \qquad ...(2)$$

Squaring and adding equations (1) and (2),

$$\frac{a^4}{x^2}\cos^2\theta + \frac{b^4}{x^2}\sin^2\theta - \frac{2a^2b^2}{xy}\sin\theta\cos\theta + \frac{a^4}{x^2}\sin^2\theta$$

$$+ \frac{b^4}{y^2}\cos^2\theta + \frac{2a^2b^2}{xy}\sin\theta\cos\theta = c^2 + 0$$

$$\Rightarrow \quad \frac{a^4}{x^2}(\cos^2\theta + \sin^2\theta) + \frac{b^4}{y^2}(\sin^2\theta + \cos^2\theta) = c^2$$

$\therefore$ $\frac{a^4}{x^2} + \frac{b^4}{y^2} = c^2$ is the required envelope.

Q13. (a) Explain the method of Lagrange multipliers. Find the minimum value of the function $f(x, y) = xy + \frac{27}{x} + \frac{27}{y}$

Answer :

Method of Lagrange's Multipliers

For answer refer Unit-III, Page No. 3.38, Q61.

Problem

Given function is,

$$f(x, y) = xy + \frac{27}{x} + \frac{27}{y} \quad \text{... (1)}$$

Differentiating equation (1) partially with respect to x,

$$\frac{\partial f}{\partial x} = y + 27\left(\frac{-1}{x^2}\right) + 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = y - \frac{27}{x^2} \quad \text{... (2)}$$

Differentiating equation (1) partially with respect to y,

$$\frac{\partial f}{\partial y} = x + 0 + 27\left(\frac{-1}{y^2}\right)$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = x - \frac{27}{y^2} \quad \text{... (3)}$$

Differentiating equation (2) partially with respect to x,

$$\frac{\partial^2 f}{\partial x^2} = 0 - 27\left(\frac{-2}{x^3}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = \frac{54}{x^3}$$

Differentiating equation (2) partially with respect to y,

$$\frac{\partial^2 f}{\partial x\,\partial y} = 1 - 0$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x\,\partial y} = 1$$

Differentiating equation (3) partially with respect to y,

$$\frac{\partial^2 f}{\partial y^2} = 0 - 27\left(\frac{-2}{y^3}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = \frac{54}{y^3}$$

The conditions for maximum or minimum value are,

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

i.e. $y - \frac{27}{x^2} = 0$ [$\because$ From equation (2)]

$$\Rightarrow \quad y = \frac{27}{x^2} \quad \text{... (4)}$$

i.e. $x - \frac{27}{y^2} = 0$ [$\because$ From equation (3)]

$$\Rightarrow \quad x = \frac{27}{y^2} \quad \text{... (5)}$$

Substituting equation (5) in equation (4),

$$y = \frac{27}{\left(27/y^2\right)^2}$$

$$\Rightarrow \quad y = \frac{y^4 . 27}{27 \times 27}$$

$$\Rightarrow \quad y = \frac{y^4}{27}$$

$\Rightarrow$ $27y - y^4 = 0$

$\Rightarrow$ $y(27 - y^3) = 0$

$\Rightarrow$ $y = 0, 27 - y^3 = 0$

$\Rightarrow$ $y = 0, 27 = y^3$

$\Rightarrow$ $y = 0, y = 3$

Substituting $y = 0$ in equation (5),

$$x = \frac{27}{0} = \infty$$

It is not possible.

Substituting $y = 3$ in equation (5),

$$x = \frac{27}{3^2} \Rightarrow x = 3$$

$\therefore$ Extreme point is (3, 3)

$$l = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(3,3)} = \frac{54}{3^3} = 2 > 0$$

$$m = \left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(3,3)} = 1$$

$$n = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(3,3)} \frac{54}{3^3} = 2$$

$ln - m^2 = (2)(2) - (1)^2$

$\Rightarrow$ $ln - m^2 = 4 - 1$

$\Rightarrow$ $ln - m^2 = 3 > 0$

$\because$ $ln - m^2 > 0$ and $l > 0$, $f(x,y)$ has minimum value at (3,3)

$$f(x, y) = 3(3) + \frac{27}{3} + \frac{27}{3} = 27$$

$\therefore$ Minimum value = 27

(b) Find the first three terms of the Taylor series of the function f(x,y) = e^x cos y around O(0,0).

Answer :

Given function is,

$f(x, y) = e^x \cos y$

$f(0, 0) = e^0 \cos(0) = 1(1) = 1$

$f_x(x, y) = e^x \cos y\ ;\ f_x(0,0) = e^0 \cos(0) = 1$

$f_y(x, y) = -e^x \sin y\ ;\ f_y(0,0) = -e^0 \sin(0) = 0$

$f_{xx}(x, y) = e^x \cos y\ ;\ f_{xx}(0,0) = e^0 \cos(0) = 1$

$f_{xy} = -e^x \sin y\ ;\ f_{xy}(0,0) = -e^0 \sin(0) = 0$

$f_{yy} = -e^x \cos y\ ;\ f_{yy}(0,0) = -e^0 \cos(0) = -1$

$f_{xxx} = e^x \cos y\ ;\ f_{xxx}(0,0) = e^0 \cos(0) = 1$

$f_{xxy} = -e^x \sin y\ ;\ f_{xxy}(0,0) = -e^0 \sin(0) = 0$

$f_{xyy} = -e^x \cos y\ ;\ f_{xyy}(0,0) = -e^0 \cos(0) = -1$

$f_{yyy} = e^x \sin y\ ;\ f_{yyy}(0,0) = e^0 \sin(0) = 0$

From Taylor's series,

$$f_x(x, y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] + [\frac{1}{3!}x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yy}(0,0)] + ...$$

Substituting the corresponding values in above equation,

$$f(x, y) = 1 + x(1) + y(0) + \frac{1}{2!}[x^2(1) + 2xy(0) + y^2(-1)] + \frac{1}{3!}[x^3(1) + 3x^2 y(0) + 3xy^2(-1) + y^3(0)] + ...$$

$$= 1 + x + 0 + \frac{1}{2}[x^2 + 0 - y^2] + \frac{1}{6}[x^3 + 0 - 3xy^2 + 0] + ...$$

$$= 1 + x + \frac{1}{2}[x^2 - y^2] + \frac{1}{6}(x^3 - 3xy^2)$$

$\therefore \quad f(x, y) = 1 + x + \frac{1}{2}[x^2 - y^2] + \frac{1}{6}(x^3 - 3xy^2).$

Q14. Evaluate $\int_{y=0}^{1}\int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx$ by changing the order of integration.

Answer :

Note : In the question, $\int_{y=0}^{1}\int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx\,dy$ is misprinted as $\int_{y=0}^{1}\int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx$

Given integral is,

$$\int_{y=0}^{1}\int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx\,dy$$

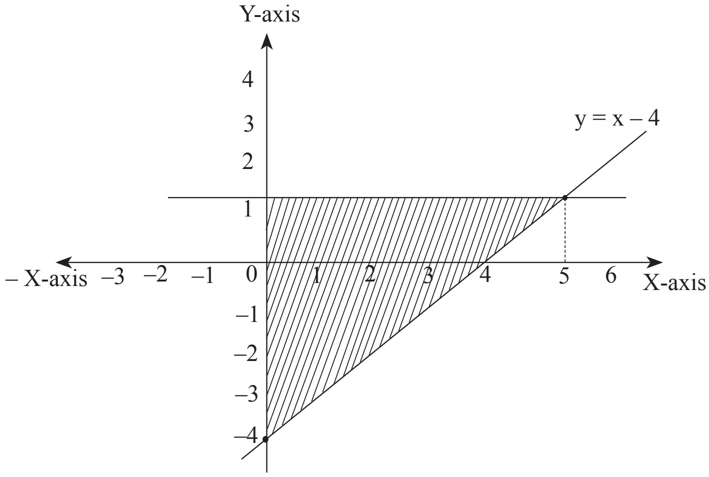
From above integral the limits of integration are, y varies from 0 to 1 and x varies from 0 to $y + 4$.

$x = y + 4$

If $y = 0$, $x = 0 + 4 = 4$ i.e. (4, 0)

If $y = 1$, $x = 1 + 4 = 5$ i.e. (5, 1)

The region of integration is shown in figure below.



From the figure, x varies from 0 to 4 and 4 to 5.

$x = y + 4 \Rightarrow y = x - 4$

$\therefore$ y varies from 0 to 1 and $x - 4$ to 1

Thus, the equivalent integral is obtained as,

$$\int_0^4 \int_0^1 \frac{2y+1}{x+1}\, dy\, dx + \int_4^5 \int_{x-4}^1 \frac{2y+1}{x+1}\, dy\, dx$$

$$= \int_0^4 \frac{1}{x+1}\left[\int_0^1 (2y+1)\, dy\right] dx + \int_4^5 \frac{1}{x+1}\left[\int_{x-4}^1 (2y+1)\, dy\right] dx$$

$$= \int_0^4 \frac{1}{x+1}\left[\frac{2y^2}{2}+y\right]_0^1 dx + \int_4^5 \frac{1}{x+1}\left[\frac{2y^2}{2}+y\right]_{x-4}^1 dx$$

$$= \int_0^4 \frac{1}{x+1}\left[y^2+y\right]_0^1 dx + \int_4^5 \frac{1}{x+1}\left[y^2+y\right]_{x-4}^1 dx$$

$$= \int_0^4 \frac{1}{x+1}\left[(1)^2 + 1 - (0)^2 - 0\right] dx + \int_4^5 \frac{1}{x+1}\left[(1)^2 + 1 - (x-4)^2 - (x-4)\right] dx$$

$$= \int_0^4 \frac{1}{x+1}\left[2 - 0\right] dx + \int_4^5 \frac{1}{x+1}\left[2 - (x^2 + 16 - 8x) - x + 4\right] dx$$

$$= \int_0^4 \frac{1}{x+1}(2)\, dx + \int_4^5 \frac{1}{x+1}(2 - x^2 - 16 + 8x - x + 4)\, dx$$

$$= 2\int_0^4 \frac{1}{x+1}\, dx + \int_4^5 \frac{-x^2 - 10 + 7x}{x+1}\, dx$$

$$= 2\left[\log(x+1)\right]_0^4 - \int_4^5 \frac{x^2 - 7x + 10}{x+1}\, dx$$

$$= 2\left[\log(4+1) - \log(0+1)\right] - \int_4^5 \frac{x^2 + x - 8x + 10}{x+1}\, dx$$

$$= 2\left[\log 5 - \log 1\right] - \int_4^5 \frac{x(x+1) - 8x + 10}{x+1}\, dx$$

$$= 2\,[\log 5 - 0] - \int_4^5 \left(\frac{x(x+1)}{x+1} - \frac{8x}{x+1} + \frac{10}{x+1} \right) dx$$

$$= 2 \log 5 - \int_4^5 \left(x - \frac{8x}{x+1} + \frac{10}{x+1} \right) dx$$

$$= 2 \log 5 - \int_4^5 x\,dx + 8 \int_4^5 \frac{x}{x+1} dx - \int_4^5 \frac{10}{x+1} dx$$

$$= 2 \log 5 - \left(\frac{x^2}{2} \right)_4^5 + 8 \int_4^5 \frac{x+1-1}{x+1}\,dx - 10 \int_4^5 \frac{1}{x+1} dx$$

$$= 2 \log 5 - \frac{1}{2}(5^2 - 4^2) + 8 \int_4^5 \frac{x+1}{x+1} dx - 8 \int_4^5 \frac{1}{x+1} dx - 10 \log \left(x+1 \right)_4^5$$

$$= 2 \log 5 - \frac{1}{2}(25 - 16) + 8 \int_4^5 1\,dx - 8 \log \left(x+1 \right)_4^5 - 10\,(\log (5+1) - \log (4+1))$$

$$= 2 \log 5 - \frac{1}{2}(9) + 8 \left(x \right)_4^5 - 8\,[\log (5+1) - \log (4+1)] - 10\,[\log 6 - \log 5]$$

$$= 2 \log 5 - \frac{9}{2} + 8(5-4) - 8\,[\log 6 - \log 5] - 10 \log 6 + 10 \log 5$$

$$= 2 \log 5 - \frac{9}{2} + 8(1) - 8 \log 6 + 8 \log 5 - 10 \log 6 + 10 \log 5$$

$$= 20 \log 5 - 18 \log 6 + \frac{7}{2}$$

$$\therefore \int_{y=0}^{1} \int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx\,dy = 20 \log 5 + \frac{7}{2} - 18 \log 6$$

Q15. Verify Gauss divergence theorem for $\overline{V} = 2xy\,\hat{i} + 6yz\hat{j} + 3zx\hat{k}$ and D is the region bounded by the coordinate planes and the plane x + y + z = 2.

Answer :

Given that,

$\overline{V} = 2xyi + 6yzj + 3zxk$

Planes are $x = 0, y = 0, z = 0, x + y + z = 2$

The region of integration is shown in figure (1).

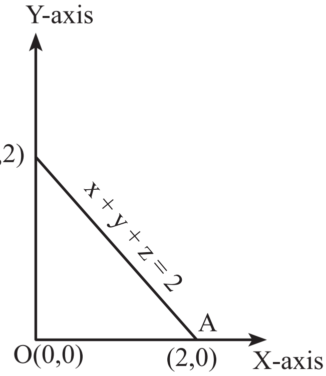


Figure (1)

By Gauss divergence theorem

$$\int\int \overline{V}.nds = \int\int_V\int \nabla.\overline{V}\ dv$$

$x + y + z = 2 \Rightarrow z = 2 - x - y$

$\therefore$ z varies from 0 to $2 - x - y$

In XY-plane, the straight line AB is given as,

$$\frac{x}{2} + \frac{y}{2} = 1$$

$\Rightarrow$ $x + y = 2$

$\Rightarrow$ $y = 2 - x$

$\therefore$ y varies from 0 to $2 - x$

From figure (2), x varies from 0 to 2

$$\iiint_V \nabla.\overline{V}\ dV$$

$$= \int\int_V\int \left(\frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(6yz) + \frac{\partial}{\partial z}(3zx)\right) dx\ dy\ dz$$

$$= \int_0^2\int_0^{2-x}\int_0^{2-x-y} (2y + 6z + 3x)\ dz\ dy\ dx = \int_0^2\int_0^{2-x}\left[\int_0^{2-x-y} (2y + 6z + 3x)\,dz\right] dy\ dx$$

$$= \int_0^2\int_0^{2-x}\left[(2yz + 6.\frac{z^2}{2} + 3xz\right]_0^{2-x-y} dy\ dx = \int_0^2\int_0^{2-x}\left[2yz + 3z^2 + 3xz\right]_0^{2-x-y} dy\ dx$$

$$= \int_0^2\int_0^{2-x} [2y(2 - x - y) + 3(2 - x - y)^2 + 3x(2 - x - y) - 2y(0) - 3(0)^2 - 3x(0)]\ dy\ dx$$

$$= \int_0^2\int_0^{2-x}\left[4y - 2xy - 2y^2 + 3(4 + x^2 + y^2 + 2xy - 4x - 4y) + 6x - 3x^2 - 3xy - 0 - 0 - 0\right] dy\ dx$$

$$= \int_0^2\int_0^{2-x}\left[4y - 2xy - 2y^2 + 12 + 3x^2 + 3y^2 + 6xy - 12x - 12y + 6x - 3x^2 - 3xy\right] dy\ dx$$

$$= \int_0^2\int_0^{2-x}\left[- 8y + xy + y^2 + 12 - 6x\right] dy\ dx$$

$$= \int_0^2\left[-\frac{8y^2}{2} + \frac{xy^2}{2} + \frac{y^3}{3} + 12y - 6xy\right]_0^{2-x} dx = \int_0^2\left[-4y^2 + \frac{xy^2}{2} + \frac{y^3}{3} + 12y - 6xy\right]_0^{2-x} dx$$

$$= \int_0^2\left[-4(2 - x)^2 + \frac{x}{2}(2 - x)^2 + \frac{(2 - x)^3}{3} + 12(2 - x) - 6x(2 - x) + 4(0)^2 - \frac{x}{2}(0)^2 - \frac{(0)^3}{3} - 12(0) + 6x(0)\right] dx$$

$$= \int_0^2\left[- 4(x^2 + 4 - 4x) + \frac{x}{2}(4 + x^2 - 4x) + \frac{1}{3}(8 - x^3 - 12x + 6x^2) + 24 - 12x - 12x + 6x^2 + 0 - 0 - 0 - 0 + 0\right]$$

$$= \int_0^2\left[- 4x^2 - 16 + 16x + \frac{4x}{2} + \frac{x^3}{2} - \frac{4x^2}{2} + \frac{8}{3} - \frac{x^3}{3} - \frac{12x}{3} + \frac{6x^2}{3} + 24 - 12x - 12x + 6x^2\right] dx$$

$$= \int_0^2\left(\frac{x^3}{6} + 2x^2 - 10x + \frac{32}{3}\right) dx = \left[\frac{x^4}{24} + \frac{2x^3}{3} - \frac{10x^2}{2} + \frac{32}{3}x\right]_0^2 = \left[\frac{x^4}{24} + \frac{2x^3}{3} - 5x^2 + \frac{32}{3}x\right]_0^2$$

$$= \frac{2^4}{24} + \frac{2(2)^3}{3} - 5(2)^2 + \frac{32}{3}(2) - \frac{0^4}{24} - \frac{2(0)^3}{3} + 5(0)^2 - \frac{32}{3}(0) = \frac{16}{24} + \frac{16}{3} - 20 + \frac{64}{3} - 0 - 0 + 0 - 0 = \frac{22}{3}$$

$$\therefore \iiint_V \nabla.\overline{V}\ dv = \frac{22}{3} \qquad ...(1)$$

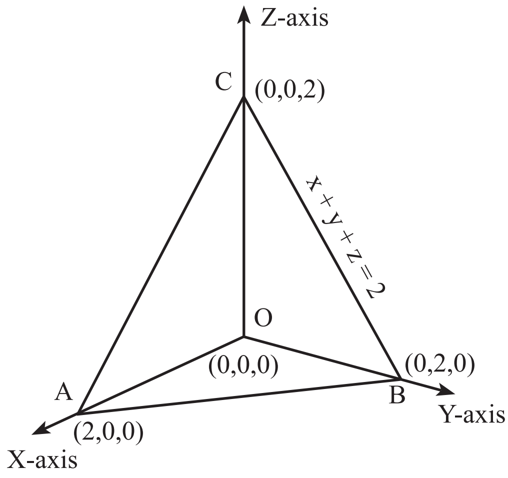


Figure (2)

Calculating the value of $\iint \bar{v}.nds$ over the four faces of the tetrahedran as shown in figure (2)

(i) For the face BOC, $x = 0$, $n = -\hat{i}$

$$\iint_{S_1} \overline{V}.\hat{n}ds = \int_0^2 \int_0^2 (2xyi + 6yzj + 3xzk).(-i)\,dy\,dz$$

$$= \int_0^2 \int_0^2 (2(0)yi + 6yzj + 3(0)zk).(-i)\,dy\,dz$$

$$= \int_0^2 \int_0^2 (0 + 6yzj + 0).(-i)\,dy\,dz$$

$$= 0$$

$$\therefore \iint_{S_1} \overline{V}.\hat{n}ds = 0$$

(ii) For the face AOC, $y = 0$, $n = -\hat{j}$

$$\iint_{S_2} \overline{V}.\hat{n}ds = \int_0^2 \int_0^2 (2xyi + 6yzj + 3xzk).(-j)\,dxdz = \int_0^2 \int_0^2 (2x(0)i + 6(0)zj + 3xzk).(-j)\,dx\,dz$$

$$= \int_0^2 \int_0^2 (0 + 0 + 3xzk).(-j)\,dx\,dz$$

$$= 0$$

$$\therefore \iint_{S_2} \overline{V}.\hat{n}ds = 0$$

(iii) For the face AOB, $z = 0$, $\hat{n} = -k$

$$\iint_{S_3} \overline{V}.\hat{n}ds = \int_0^2 \int_0^2 (2xyi + 6yzj + 3xzk).(-k)\,dxdy$$

$$= \int_0^2 \int_0^2 (2xyi + 6y(0)j + 3(0)xk).(-k)\,dx\,dy$$

$$= \int_0^2 \int_0^2 (2xyi + 0 + 0).(-k)\,dx\,dy$$

$$= 0$$

$$\therefore \iint_{S_3} \overline{V}.\hat{n}ds = 0$$

(iv) For the face ABC, $z = 2 - x - y$, $\hat{n} = i + j + k$

$$\iint_{S_4} \overline{V}.\hat{n}ds = \int_0^2 \int_0^{2-x} (2xyi + 6yzj + 3xzk).(i + j + k)\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (2xy + 6yz + 3zx)\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (2xy + 6y(2 - x - y) + 3x(2 - x - y))\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (2xy + 12y - 6xy - 6y^2 + 6x - 3x^2 - 3xy))\, dy\, dx$$

$$= \int_0^2 \left[\int_0^{2-x} (-3x^2 - 7xy + 12y + 6x - 6y^2)\, dy\, dx\right]$$

$$= \int_0^2 \left[(-3x^2y - \frac{7xy^2}{2} + \frac{12y^2}{2} + 6xy - \frac{6y^3}{3}\right]_0^{2-x} dx$$

$$= \int_0^2 \left[(-3x^2y - \frac{7xy^2}{2} + 6y^2 + 6xy - 2y^3\right]_0^{2-x} dx$$

$$= \int_0^2 \left[(-3x^2(2-x) - \frac{7x}{2}(2-x)^2 + 6(2-x)^2 + 6x(2-x) - 2(2-x)^3 + 3x^2(0) + \frac{7x}{2}(0)^2 - 6(0)^2 - 6x(0) + 12(0)^3\right] dx$$

$$= \int_0^2 \left[-6x^2 + 3x^3 - \frac{7x}{2}(x^2 + 4 - 4x) + 6(x^2 + 4 - 4x) + 12x - 6x^2 - 2(-x^3 + 8 - 12x + 6x^2) + 0 + 0 - 0 - 0 + 0\right] dx$$

$$= \int_0^2 \left[-6x^2 + 3x^3 - \frac{7x^3}{2} - \frac{28x}{2} + \frac{28x^2}{2} + 6x^2 + 24 - 24x + 12x - 6x^2 + 2x^3 - 16 + 24x - 12x^2\right] dx$$

$$= \int_0^2 \left(\frac{3x^3}{2} - 4x^2 - 2x + 8\right) dx$$

$$= \left[\frac{3}{2}\left(\frac{x^4}{4}\right) - 4\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right) + 8x\right]_0^2$$

$$= \left[\frac{3x^4}{8} - \frac{4x^3}{3} - x^2 + 8x\right]_0^2$$

$$= \frac{3}{8}(2)^4 - \frac{4}{3}(2)^3 - (2)^2 + 8(2) - \frac{3}{8}(0)^4 + \frac{4}{3}(0)^3 + (0)^2 - 8(0)$$

$$= \frac{3}{8}(16) - \frac{4}{3}(8) - 4 + 16 - 0 + 0 + 0 - 0$$

$$= \frac{22}{3}$$

$$\iint \overline{V}.\hat{n}ds = \iint_{S_1} \overline{V}.\hat{n}ds + \iint_{S_2} \overline{V}.\hat{n}ds + \iint_{S_3} \overline{V}.\hat{n}ds + \iint_{S_4} \overline{V}.\hat{n}ds$$

Substituting the corresponding values in above equation,

$$\iint \overline{V}.\hat{n}ds = 0 + 0 + 0 + \frac{22}{3}$$

$$\Rightarrow \quad \iint \overline{V}.\hat{n}ds = \frac{22}{3} \qquad \text{...(2)}$$

From equations (1) and (2),

$$\iint \overline{V}.\hat{n}ds = \iiint_V \nabla.\overline{V}.dv$$

Hence, Gauss divergence theorem is verified.

Q16. (a) Find the extreme values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 = 1$.

Answer :

Given function is,

$f(x,y) = x^2 + 3y^2 + 2y$ ... (1)

$\phi(x,y) = x^2 + y^2 = 1$

$\Rightarrow \phi(x,y) = x^2 + y^2 - 1$ ... (2)

According to Lagrange's function,

$F(x,y) = f(x,y) + \lambda\, \phi(x,y)$ ... (3)

Where,

λ - Lagrange's multiplier

Substituting equations (1) and (2) in equation (3),

$F(x,y) = x^2 + 3y^2 + 2y + \lambda\,(x^2 + y^2 - 1)$ ... (4)

Differentiating equation (4) partially with respect to x,

$\frac{\partial F}{\partial x} = 2x + 0 + 0 + \lambda\,(2x + 0 - 0)$

$\Rightarrow \frac{\partial F}{\partial x} = 2x + \lambda\, 2x$ ... (5)

Differentiating equation (4) partially with respect to y,

$\frac{\partial F}{\partial y} = 0 + 6y + 2 + \lambda\,(0 + 2y - 0)$

$\Rightarrow \frac{\partial F}{\partial y} = 6y + 2 + 2\lambda x$ ... (6)

For maximum or minimum value, $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0$

From equation (5),

$2x + 2\lambda x = 0$

$\Rightarrow 2x\,(1 + \lambda) = 0$

$\Rightarrow 2x = 0,\ 1 + \lambda = 0$

$\Rightarrow x = 0,\ \lambda = -1$

From equation (6),

$6y + 2 + 2\lambda y = 0$

$\Rightarrow 3y + 1 + \lambda y = 0$

$\Rightarrow (3 + \lambda)y = -1$

$\Rightarrow y = \frac{-1}{\lambda + 3}$

$\Rightarrow y = \frac{-1}{-1+3}$ $[\because \lambda = -1]$

$\Rightarrow y = \frac{-1}{2}$

Substituting $x = 0$ in equation (2),

$x^2 + y^2 - 1 = 0$

$\Rightarrow 0^2 + y^2 = 1$

$\Rightarrow y = \pm 1$

Substituting $y = \frac{-1}{2}$ in equation (2),

$x^2 + y^2 - 1 = 0$

$\Rightarrow x^2 + \left(\frac{-1}{2}\right)^2 - 1 = 0$

$\Rightarrow x^2 + \frac{1}{4} - 1 = 0$

$\Rightarrow x^2 = \frac{3}{4}$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2}$

$\therefore$ The extreme points are $\left(0, \frac{-1}{2}\right)$, (0, 1), (0, – 1), $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$, $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$

$f(x,y)\Big|_{\left(0,\frac{-1}{2}\right)} = 0^2 + 3\left(\frac{-1}{2}\right)^2 + 2\left(\frac{-1}{2}\right) = \frac{-1}{4}$

$f(x,y)\Big|_{(0,1)} = 0^2 + 3(1)^2 + 2(1) = 5$

$f(x,y)\Big|_{(0,-1)} = 0^2 + 3(-1)^2 + 2(-1) = 1$

$f(x,y)\Big|_{\left(\frac{\sqrt{3}}{2},\frac{-1}{2}\right)} = \left(\frac{\sqrt{3}}{2}\right)^2 + 3\left(\frac{-1}{2}\right)^2 + 2\left(\frac{-1}{2}\right) = \frac{1}{2}$

$f(x,y)\Big|_{\left(\frac{-\sqrt{3}}{2},\frac{-1}{2}\right)} = \left(\frac{-\sqrt{3}}{2}\right)^2 + 3\left(\frac{-1}{2}\right)^2 + 2\left(\frac{-1}{2}\right) = \frac{1}{2}$

$\therefore$ Maximum value is 5 occurs at (0, 1) and minimum value is $\frac{-1}{4}$ occurs at $\left(0, \frac{-1}{2}\right)$.

(b) Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer :

For answer refer Unit-II, Q59, Page No. 2.30.

Q17. (a) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$ by changing to polar coordinates.

Answer :

Given integral is,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

From the above limits, y varies from 0 to $\sqrt{1-x^2}$ and x varies from 0 to 1.

i.e. $y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1 - x^2$

$\Rightarrow x^2 + y^2 = 1$

$(r\cos\theta)^2 + (r\sin\theta)^2 = 1$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1$

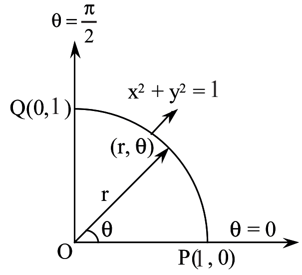
$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1$

$\Rightarrow r^2(1) = 1$

$\Rightarrow r^2 = 1$

$\Rightarrow r = 1$

The region of integration is the first quadrant of a circle $x^2 + y^2 = 1$ represented by the region *OPQ* as shown in figure below.



Figure

The limits of θ are 0 to $\frac{\pi}{2}$ and limits of *r* are 0 to 1.

$$\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^{\pi/2}\int_0^1 r^2 . r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{\pi/2}\frac{1}{4}\left(r^4\right)_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(1^4 - 0^4\right)d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 1d\theta$$

$$= \frac{1}{4}\left(\theta\right)_0^{\pi/2}$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{8}$$

$$\therefore \int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \frac{\pi}{8}$$

(b) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$, $z + 3 = x^2 + y^2$ at (–2, 1, 2).

Answer :

Given surfaces are,

$$\phi_1 = x^2 + y^2 + z^2 = 9$$

$$\Rightarrow \phi_1 = x^2 + y^2 + z^2 - 9$$

$$\phi_2 = z + 3 = x^2 + y^2$$

$$\Rightarrow \phi_2 = x^2 + y^2 - z - 3$$

Point $p = (-2, 1, 2)$

Let $\overline{n_1}$ and $\overline{n_2}$ be the normals to the surfaces at (– 2, 1, 2)

Where,

$$\overline{n}_1 = \nabla\phi_1$$

$$= \bar{i}\frac{\partial}{\partial x}\phi_1 + \bar{j}\frac{\partial}{\partial y}\phi_1 + \bar{k}\frac{\partial}{\partial z}\phi_1 \qquad \left[\because \nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right]$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$$

$$\overline{n_1}\Big|_{(-2,1,2)} = -4\bar{i} + 2\bar{j} + 4\bar{k}$$

And $\overline{n_2} = \nabla\phi_2$

$$= \bar{i}\frac{\partial}{\partial x}\phi_2 + \bar{j}\frac{\partial}{\partial y}\phi_2 + \bar{k}\frac{\partial}{\partial z}\phi_2$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(-1)$$

$$\overline{n_2}\Big|_{(-2,1,2)} = -4\bar{i} + 2\bar{j} - \bar{k}$$

Angle between the surfaces is given as,

$$\cos\theta = \frac{\overline{n}_1 . \overline{n}_2}{|\overline{n}_1||\overline{n}_2|}$$

$$= \frac{(-4\bar{i} + 2\bar{j} + 4\bar{k})(-4\bar{i} + 2\bar{j} - \bar{k})}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}}$$

$$= \frac{16}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \cos\theta = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$